||Jai Sri Gurudev||

S. J. C. INSTITUTE OF TECHNOLOGY, CHICKBALLAPUR

Department of Mathematics

LECTURE NOTES

MATHEMATICS-1 FOR MECHNICALENGINEERING STREAM (22MATM11)

MODULE-4

DIFFERENTIAL EQUATIONS OF HIGHER ORDER &FIRST DEGREE

Prepared By:

Purushotham.P
Assistant Professor
SJC Institute of Technology
Email id: purushotham@sjcit.ac.in
Phone: 7338005244

Introduction :-

Let 'y' be a function in the variable of x, then the equation is

$$a_0 \frac{d^n y}{dx^n} + a_1 \frac{d^{n-1} y}{dx^{n-1}} + a_2 \frac{d^{n-2} y}{dx^{n-2}} + \text{----} a_n y = Q(x) \quad \text{——(1)}$$

is called the linear differential equation of nth order and first degree.

Let $\frac{d}{dx} = D$, which is called the differential operator

$\therefore$ (1) $\Rightarrow a_0 D^n y + a_1 D^{n-1} + a_2 D^{n-2} + \text{----} a_n y = Q(x)$

$\Rightarrow (a_0 D^n + a_1 D^{n-1} + a_2 D^{n-2}) y$

$\Rightarrow f(D) y = Q(x)$

can represent a polynomial in 'D' of degree 'n'.

Equation (1) follows :-

1. If $a_0, a_1, a_2, a_3 --- a_n$ are constants and $Q(x)$ is 0, then eqn (1) can be called as the linear homogenous differential equations with constant co-efficient.

2. If $a_0, a_1, a_2 --- a_n$ are constants and $Q(x) \neq 0$, then eqn(1) can be called as the linear non-homogenous D.E with constant co-efficients.

3. If $a_0, a_1, a_2 --- a_n$ are the functions in the variable 'x' where $Q(x)=0$, then eqn (1) can be called as the linear homogenous D.E with variable co-efficient.

4. If $a_0, a_1, a_2 --- a_n$ are the function in the variable 'x' where $Q(x) \neq 0$, then eqn (1) can be called as linear homogenous D.E with variable co-efficient.

Solution of Linear Differential Equations with constant co-efficient.

1. Write given differential equation in the notations of $y, y', y'', y''' ----$.

2. Let, $D = \frac{d}{dx}$ as the differential operator and rewrite the given D.E in the form of $f(D)y = Q(x)$.

3. Write the solution of the given D.E as

$y = CF + PI$ (or) $y = y_c + y_p$ where,

CF (or) y_c = Complementary function. and

PI (or) y_p = the particular integral,

which is depending on $Q(x)$

$$\therefore \; y_p = \frac{Q(x)}{f(D)}$$

Evaluation of Complementary Function :-

1. Identify the polynomial $f(D)$ from the given D.E.

2. Write an auxillary equation $f(m)=0$, which is a polynomial and can provide 'n' no of roots, they are, $m_1, m_2, m_3 \ldots m$ (say)

then the complementary function is followed as ;

a. If all the given roots are real and distinct (or) real and different, then

$$y_c = c_1 e^{m_1 x} + c_2 e^{m_2 x} + c_3 e^{m_3 x} \ldots + c_n e^{m_n x}$$

Example :- $m = 1, -1, 3$

$$y_c = c_1 e^{x} + c_2 e^{-x} + c_3 e^{3x}$$

b. If the first two roots are equal and remaining are real and distinct, then

$$y_c = (c_1 + c_2 x) e^{m_1 x} + c_3 e^{m_3 x} + c_4 e^{m_4 x} + \ldots c_n e^{m_n x}$$

Example :- $m = 2, 2, 5, 6$

$$y_c = (c_1 + c_2 x) e^{2x} + c_3 e^{5x} + c_4 e^{6x}$$

c. If the first three roots are equal and remaining (or) real and distinct.

$$y_c = (c_1 + c_2 x + c_3 x^2) e^{m_1 x} + c_4 e^{m_4 x} \text{----} c_n e^{m_n x}$$

Example :- $m = -1, -1, -1, 3, 2$

$$y_c = (c_1 + c_2 x + c_3 x^2) e^{-x} + c_4 e^{3x} + c_5 e^{2x}$$

d. If first 2 roots are complex and remaining are real and distinct.

$\therefore \; m_1 + i m_2, m_3, m_4 \text{----} m_n$

$$y_c = (c_1 \cos m_2 x + c_2 \sin m_2 x) e^{m_1 x} + c_3 e^{m_3 x} + c_4 e^{m_4 x} + \text{----} c_n e^{m_n x}$$

Example :- $2+3i, -1, 4$

$$y_c = (c_1 \cos 3x + c_2 \sin 3x) e^{2x} + c_3 e^{-x} + c_4 e^{4x}$$

Evaluation of Particular integral

1. If $Q(x) = e^{ax}$, then

$$y_p = \frac{e^{ax}}{f(D)} = \frac{e^{ax}}{f(a)}, \quad f(a) \neq 0$$

$$y_p = \frac{e^{ax}}{f(D)} = \frac{e^{ax}}{(D-a)^k \phi(D)} = \frac{x^k}{k!} \frac{e^{ax}}{\phi(D)}$$

$$= \frac{x^k}{k!} \frac{e^{ax}}{\phi(a)}, \quad f(a) = 0$$

Example :-

i.) $\dfrac{e^{-x}}{D^2+D-2} = \dfrac{e^{-x}}{(-1)^2-1-2}$

$= \dfrac{e^{-x}}{1-1-2}$

$= \dfrac{e^{-x}}{-2}$

$= -\dfrac{e^{-x}}{2}$

ii.) $\dfrac{e^{-x}}{D^2-1} = \dfrac{e^{-x}}{(D+1)^1(D-1)}$

$= \dfrac{x^1}{1!}\dfrac{e^{-x}}{D-1}$

$= \dfrac{x^1}{1!}\dfrac{e^{-x}}{(-1-1)}$

$= \dfrac{-xe^{-x}}{2}$

2. If $Q(x) = \cos ax$, then $y_p = \dfrac{\cos ax}{f(D)}$, here replace D^2 as $-a^2$ and when $f(-a^2) \neq 0$ and $\dfrac{\cos ax}{D^2+a^2} = \dfrac{x}{2a}\sin ax$ and it is same for $Q(x) = \sin ax$, $\dfrac{\sin ax}{D^2+a^2} = -\dfrac{x}{2a}\cos ax$

Example :- $y_p = \dfrac{\cos 2x}{D^2+4D-2}$, $a=2$

$= \dfrac{\cos 2x}{-2^2-4D-2}$

$= \dfrac{\cos 2x}{-4+4D-2}$

$= \dfrac{\cos 2x}{4D-6}$

$= \dfrac{(4D+6)\cos 2x}{(4D-6)(4D+6)}$

$$= \frac{(4D+6)\cos 2x}{16D^2 - 36}$$

$$= \frac{(4D+6)\cos 2x}{16(-2)^2 - 36}$$

$$= \frac{(4D+6)\cos 2x}{-64-36}$$

$$= -\frac{1}{100}[4D+6]\cos 2x$$

$$= -\frac{1}{100}[4D(\cos 2x) + 6\cos 2x]$$

$$= -\frac{1}{100}[4(-2\sin 2x) + 6\cos 2x]$$

$$= -\frac{1}{100}[-8\sin 2x + 6\cos 2x]$$

$$= \frac{1}{100}[8\sin 2x - 6\cos 2x]$$

(ii) $$y_p = \frac{\sin 3x}{D^2+9}$$

$$= \frac{-x}{2(3)}\cos 3x$$

$$= \frac{x}{6}\cos 3x$$

3. If $Q(x) = \phi(x)$, where $\phi(x)$ is a polynomial in x, then $y_p = \frac{\phi(x)}{f(D)}$

write $f(D) = [1 \pm \phi(D)]^k$

$f(D) = \lambda[1 \pm \phi(D)]^k$

$$\therefore \quad y_p = \frac{\phi(x)}{\lambda[1 \pm \phi(D)]^{K}}$$

$$\Rightarrow \quad y_p = \frac{1}{\lambda}[1 \pm \phi(D)]^{-K}\,\phi(x)$$

and it can be solved by using formulas

$$(1-x)^{-1} = 1 + x + x^2 + x^3 + \cdots$$

$$(1+x)^{-1} = 1 - x + x^2 - x^3 + \cdots$$

$$(1-x)^{-2} = 1 + 2x + 3x^2 + 4x^3 + \cdots$$

$$(1+x)^{-2} = 1 - 2x + 3x^2 - 4x^3 + \cdots$$

Example :- (1) $y_p = \dfrac{2x+1}{D^2+2}$

$$= \frac{1}{2}\,\frac{2x+1}{\left(1+\frac{D^2}{2}\right)}$$

$$= \frac{1}{2}\left[1+\frac{D^2}{2}\right]^{-1}[2x+1]$$

$$= \frac{1}{2}\left[1-\frac{D^2}{2}+\left(\frac{D^2}{2}\right)^2-\left(\frac{D^2}{2}\right)^3 \cdots\right][2x+1]$$

$$= \frac{1}{2}\left[(2x+1)-\frac{1}{2}D^2(2x+1)+\frac{1}{4}D^4(2x+1)\right]$$

$$= \frac{1}{2}\left[(2x+1)-\frac{1}{2}(0)\right]$$

$$y_p = \frac{1}{2}[2x+1]$$

1. Solve $y'' + 3y' + 2y = 0$

$\Rightarrow$ Given :-

$$y'' + 3y' + 2y = 0$$

$$\Rightarrow (D^2 + 3D + 2)\, y = 0 \quad , \quad D = \frac{d}{x}$$

$$\Rightarrow f(D)y = 0$$

where, $f(D) = D^2 + 3D + 2$

$\therefore$ The AE is

$$f(m) = 0$$

$$\Rightarrow m^2 + 3m + 2 = 0$$

$$\Rightarrow m^2 + m + 2m + 2 = 0$$

$$\Rightarrow m(m+1) + 2(m+1) = 0$$

$$\Rightarrow (m+1)(m+2) = 0$$

$$\Rightarrow m+1 = 0 \quad , \quad m+2 = 0$$

$$\Rightarrow m = -1 \quad , \quad m = -2$$

$$\therefore y_c = y = c_1 e^{-x} + c_2 e^{-2x}$$

2. Solve $\frac{d^3y}{dx^3} + 2\frac{d^2y}{dx^2} + \frac{dy}{dx} = 0$

$\Rightarrow$ Given :- $\frac{d^3y}{dx^3} + 2\frac{d^2y}{dx^2} + \frac{dy}{dx} = 0$

$$\Rightarrow D^3 + 2D^2 + Dy = 0$$

$$\Rightarrow f(D)y = 0$$

where, $f(D) = D^3 + 2D^2 + D$

$\therefore$ The AE is

$$f(m)=0$$

$$\Rightarrow m^3+2m^2+m=0$$

$$\Rightarrow m(m^2+2m+1)=0$$

$$\Rightarrow m=0, \quad m^2+2m+1=0$$

$$\Rightarrow m=0, \quad (m+1)^2=0$$

$$\Rightarrow m=0, \quad (m+1)(m+1)=0$$

$$\Rightarrow m=0, \quad m=-1, \quad m=-1$$

$$\therefore y = c_1e^{0x} + (c_2+c_3x)e^{-x}$$

$$\Rightarrow y = c_1 + (c_2+c_3x)e^{-x}$$

3. Solve $(4D^4-8D^3-7D^2+11D+6)y=0$

$\Rightarrow$ Given,

$$(4D^4-8D^3-7D^2+11D+6)y=0$$

$$\Rightarrow f(D)y=0$$

$\therefore$ The AE is

$$f(m)=0$$

$$\Rightarrow 4m^4-8m^3-7m^2+11m+6=0$$

$$\Rightarrow (m+1)(4m^3-12m^2+5m+6)=0$$

$$\Rightarrow m+1=0, \quad 4m^3-12m^2+5m+6=0$$

$$\Rightarrow m=-1, \quad (m-2)(4m^2-4m-3)=0$$

$$\Rightarrow m=-1, \quad m=2, \quad 4m^2-4m-3=0$$

$$\Rightarrow 2m(2m+1)-3(2m+1)=0$$

$$\Rightarrow (2m+1)(2m-3)=0$$

$\Rightarrow 2m+1=0 \quad , \quad 2m-3=0$

$\Rightarrow \quad m=-1 \, , \quad m=2 \quad , \quad m=-\frac{1}{2} \, , \, m=\frac{3}{2}$

$\therefore \quad y = c_1 e^{-x} + c_2 e^{2x} + c_3 e^{-x/2} + c_4 e^{3/2}$

4. Solve $(D^4+4D^3-5D^2-36D-36)y=0$

$\Rightarrow$ Given:-

$$(D^4+4D^3-5D^2-36D-36)y=0$$

$$\Rightarrow f(D)\,y=0$$

$\therefore$ The AE is

$$f(m)=0$$

$$\Rightarrow m^4+4m^3-5m^2-36m-36=0$$

$$\Rightarrow (m+2)(m^3+2m^2-9m-18)=0$$

$$\Rightarrow (m+2)(m+2)(m^2-9)=0$$

$$\Rightarrow (m+2)(m+2)(m-3)(m+3)=0$$

$$\Rightarrow m=-2, \quad m=-2, \quad m=3, \quad m=-3$$

$$\therefore y_c = (c_1+c_2x)e^{-2x} + c_3e^{3x} + c_4e^{-3x}$$

5. Solve $(4D^4-4D^3-23D^2+12D+36)y=0$

$\Rightarrow$ Given:-

$$(4D^4-4D^3-23D^2+12D+36)y=0$$

$$\Rightarrow f(D)y=0$$

$\therefore$ The A.E is

$$f(m)=0$$

$$\Rightarrow 4m^4-4m^3-23m^2+12m+36=0$$

$\Rightarrow (m-2)(4m^3+4m^2-15m-18)=0$

$\Rightarrow (m-2)(m-2)(4m^2+12m+9)=0$

$\Rightarrow (m-2)(m-2)(4m^2+6m+6m+9)=0$

$\Rightarrow (m-2)(m-2)\ 2m(2m+3)+3(2m+3)=0$

$\Rightarrow (m-2)(m-2)(2m+3)(2m+2)=0$

$\Rightarrow m=2,\ 2,\ -3/2,\ -3/2$

$\therefore\ y=(c_1+c_2x)e^{2x}+(c_3+c_4x)e^{-3/2x}$

6. Solve $(D^4-1)y=0$

$\Rightarrow$ Given:-

$(D^4-1)y=0$

$\Rightarrow f(D)y=0$

$\therefore$ The AE is

$f(m)=0$

$\Rightarrow m^4-1=0$

$\Rightarrow (m^2)^2-1^2=0$

$\Rightarrow (m^2-1)(m^2+1)=0$

$\Rightarrow m^2-1=0\ ,\ m^2+1=0$

$\Rightarrow m^2=1\ ,\ m^2=-1$

$\Rightarrow m=\pm 1 \quad m=\pm i$

$\Rightarrow m=-1,1 \quad m=0\pm 1i$

$\therefore\ y=c_1e^{-x}+c_2e^{x}+(c_3\cos x+c_4\sin x)e^{0x}$

$\Rightarrow y=c_1e^{-x}+c_2e^{x}+(c_3\cos x+c_4\sin x)$

7. Solve $(D^3 - 6D^2 + 11D - 6)y = e^{2x}$

$\Rightarrow$ Given:-

$$(D^3 - 6D^2 + 11D - 6)\, y = e^{2x}$$

$$\Rightarrow f(D)y = e^{2x}$$

$\therefore$ The AE is

$$f(m) = 0$$

$$\Rightarrow m^3 - 6m^2 + 11m - 6 = 0$$

$$\Rightarrow (m-1)\,(m^2 - 5m + 6) = 0$$

$$\Rightarrow m = 1\,,\ (m^2 - 5m + 6) = 0$$

$$\Rightarrow m = 1\,,\ m^2 - 2m - 3m + 6 = 0$$

$$\Rightarrow m = 1\,,\ m(m-2) - 3(m-2) = 0$$

$$\Rightarrow m = 1\,,\ (m-2)\,(m-3) = 0$$

$$\Rightarrow m = 1\,,\ m = 2\,,\ m = 3$$

$$\therefore y_c = c_1 e^{x} + c_2 e^{2x} + c_3 e^{3x}$$

$$\therefore y_p = \frac{e^{2x}}{f(D)}$$

$$\Rightarrow y_p = \frac{e^{2x}}{D^3 - 6D^2 + 11D - 6}$$

$$= \frac{e^{2x}}{(D-2)(D-1)(D-3)}$$

$$= \frac{e^{2x}}{(D-2)'(D^2 - 4D + 3)}$$

$$= \frac{x^1}{1!} \frac{e^{2x}}{[2^2-4(2)+3]}$$

$$= \frac{x.e^{2x}}{4-8+3}$$

$$= \frac{x.e^{2x}}{-1}$$

$$\Rightarrow y_p = -x.e^{2x}$$

$\therefore$ The solution is $y = y_c + y_p$

$$\therefore y = c_1e^x + c_2e^{2x} + c_3e^{3x} - xe^{2x}$$

8. Solve $(D^3+2D^2+D)y = e^{-x}$

$\Rightarrow$ Given :-

$$(D^3+2D^2+D)y = e^{-x}$$

$$\Rightarrow f(D)y = e^{-x}$$

$\therefore$ The AE is

$$f(m)=0$$

$$\Rightarrow m^3+2m^2+m=0$$

$$\Rightarrow m(m^2+2m+1)=0$$

$$\Rightarrow m(m+1)^2=0$$

$$\Rightarrow m=0,\ (m+1)(m+1)=0$$

$$\Rightarrow m=0,-1,-1$$

$$\therefore y_c = c_1 + (c_2+c_3x)e^{-x}$$

$$\therefore y_p = \frac{e^{-x}}{f(D)}$$

$$\Rightarrow y_p = \frac{e^{-x}}{D^3+2D^2+D}$$

$$\Rightarrow y_p = \frac{e^{-x}}{D(D+1)^2}$$

$$\Rightarrow y_p = \frac{x^2}{2!}\frac{e^{-x}}{(-1)}$$

$$\Rightarrow y_p = \frac{-x^2 . e^{-x}}{2}$$

$\therefore$ The solution is , $y = y_c + y_p$

$$\Rightarrow y = c_1 + (c_2 + c_3 x)e^{-x} - \frac{x^2}{2}. e^{-x}$$

9. Solve $(D^2 + 4D + 3)y = e^{-x}$

$\Rightarrow$ Given :- $(D^2 + 4D + 3)y = e^{-x}$

$\Rightarrow f(D)y = e^{-x}$

$\therefore$ The A.E is

$f(m) = 0$

$\Rightarrow m^2 + 4m + 3 = 0$

$\Rightarrow m^2 + m + 3m + 3 = 0$

$\Rightarrow (m+1)(m+3) = 0$

$\Rightarrow m = -1, -3$

$\therefore y_c = c_1 e^{-x} + c_2 e^{-3x}$

$$\therefore y_p = \frac{e^{-x}}{f(D)}$$

$$\Rightarrow \quad y_p = \frac{e^{-x}}{D^2+4D+3}$$

$$\Rightarrow \quad y_p = \frac{e^{-x}}{(D+1)(D+3)}$$

$$\Rightarrow \quad y_p = \frac{x}{1!}\,\frac{e^{-x}}{[-1+3]}$$

$$\Rightarrow \quad y_p = \frac{x.e^{-x}}{2}$$

$\therefore$ The solution is $y = y_c + y_p$

$$\therefore \quad y = c_1 e^{-x} + c_2 e^{-3x} + \frac{x.e^{-x}}{2}$$

10. Solve $(D^2-6D+9)y = 6e^{3x} + 7e^{-2x} - \log 2$

$\Rightarrow$ Given :-

$$(D^2-6D+9)y = 0$$

$$\Rightarrow \quad f(D)y = 0$$

$\therefore$ The AE is $f(m)=0$

$$\Rightarrow \quad m^2-6m+9=0$$

$$\Rightarrow \quad m^2-3m-3m+9=0$$

$$\Rightarrow \quad m(m-3)-3(m-3)=0$$

$$\Rightarrow \quad (m-3)(m-3)=0$$

$$\Rightarrow \quad m=3,\ 3$$

$$\therefore \quad y_c = (c_1+c_2x)e^{3x}$$

$$\therefore \quad y_p = \frac{6e^{3x}+7e^{-2x}-\log 2}{f(D)}$$

$$= \frac{6e^{3x}}{f(D)} + \frac{7e^{-2x}}{f(D)} - (\log 2)\frac{e^{0.x}}{f(D)}$$

$$= \frac{6e^{3x}}{(D-3)^2} + 7.\frac{e^{-2x}}{(D-3)^2} - \log 2 \frac{e^{0.x}}{(D-3)^2}$$

$$= \frac{6x^2}{2!}\frac{e^{3x}}{1} + \frac{7e^{-2x}}{(-2-3)^2} - \log 2 \frac{e^{0x}}{(0-3)^2}$$

$$\Rightarrow \quad y_P = 3x^2 e^{3x} + \frac{7}{25}e^{-2x} - \frac{\log 2}{9}$$

$\therefore$ The solution is $y = y_c + y_P$

$$\Rightarrow \quad y = (c_1 + c_2 x)e^{3x} + 3x^2 e^{3x} + \frac{7}{25}e^{-2x} - \frac{\log 2}{9}$$

11. Solve $\dfrac{d^2y}{dx^2} - 4y = \cosh(2x-1) + 3^x$

$\Rightarrow$ Given:- $\dfrac{d^2y}{dx^2} - 4y = \cosh(2x-1) + 3^x$

$$\Rightarrow \quad (D^2-4)y = \cosh(2x-1) + 3^x$$

$$\Rightarrow \quad f(D)y = \cosh(2x-1) + 3^x$$

$\therefore$ The AE is

$$f(m) = 0$$

$$\Rightarrow m^2 - 4 = 0$$

$$\Rightarrow (m-2)(m+2) = 0$$

$$\Rightarrow m = 2, -2$$

$$\therefore \quad y_c = c_1 e^{2x} + c_2 e^{-2x}$$

$$\therefore\ y_p = \frac{\cosh(2x-1)+3^x}{f(D)}$$

$$\Rightarrow\ y_p = \frac{\dfrac{e^{2x-1}+e^{-2x-1}}{2}}{D^2 \mp 4} + \frac{e^{(\log_e 3)x}}{D^2-4}$$

$$\Rightarrow\ y_p = \frac{1}{2}\frac{e^{2x-1}}{(D-2)(D+2)} + \frac{1}{2}\frac{e^{-2x+1}}{(D+2)(D-2)} + \frac{e^{(\log 3)x}}{D^2-4}$$

$$\Rightarrow\ y_p = \frac{1}{2}\frac{x}{1!}\frac{e^{2x-1}}{(2+2)} + \frac{1}{2}\frac{x}{1!}\frac{e^{-2x+1}}{(-2-2)} + \frac{e^{(\log 3)x}}{(\log 3)^2-4}$$

$$\Rightarrow\ y_p = \frac{x}{4}\frac{e^{2x-1}}{2} - \frac{x}{4}\frac{e^{-(2x-1)}}{2} + \frac{3^x}{(\log 3)^2-4}$$

$$\Rightarrow\ y_p = \frac{x}{2}\sinh(2x-1) + \frac{3^x}{(\log 3)^2-4}$$

12. Solve $\dfrac{d^2y}{dx^2} + y = \cos\left[\dfrac{\pi}{2} - x\right] + e^x$

$\Rightarrow$ Given :-

$\Rightarrow\ (D^2+1)y = \sin x + e^x$

$\Rightarrow\ f(D)y = \sin x + e^x$

$\therefore$ The AE is

$f(m)=0$

$\Rightarrow\ m^2+1 = 0$

$\Rightarrow\ m^2 = -1$

$\Rightarrow\ m = \pm i$

$\Rightarrow\ m = 0 \pm 1i$

$$\therefore \quad y_c = (c_1 \cos x + c_2 \sin x)e^{0.x}$$

$$y_c = c_1 \cos x + c_2 \sin x$$

$$\therefore \quad y_p = \frac{\sin x + e^x}{f(D)}$$

$$\Rightarrow \quad y_p = \frac{\sin x}{f(D)} + \frac{e^x}{f(D)}$$

$$\Rightarrow \quad y_p = \frac{\sin x}{D^2+1} + \frac{e^x}{D^2+1}$$

$$\Rightarrow \quad y_p = -\frac{x}{2(1)} \cos x + \frac{e^x}{1+1}$$

$$\Rightarrow \quad y_p = \frac{-x}{2} \cos x + \frac{1}{2} e^x$$

$\therefore$ The solution is $y = y_c + y_p$

$$\Rightarrow \quad y = c_1 \cos x + c_2 \sin x - \frac{x}{2} \cos x + \frac{1}{2} e^x$$

13. Solve $(D^3 - 1)\, y = 3 \cos 2x$

$\Rightarrow$ Given:-

$$(D^3 - 1)\, y = 3 \cos 2x$$

$\therefore$ The AE is $f(m) = 0$

$$\Rightarrow \quad m^3 - 1 = 0$$

$$\Rightarrow \quad (m-1)(m^2 + m + 1) = 0$$

$$\Rightarrow \quad m - 1 = 0, \quad m^2 + m + 1 = 0$$

$$\Rightarrow \quad m = 1, \quad m = \frac{-1 \pm \sqrt{(1)^2 - 4(1)(1)}}{2(1)}$$

$$\Rightarrow \quad m=1, \quad m=\frac{-1\pm\sqrt{-3}}{2}$$

$$\Rightarrow \quad m=1, \quad m=\frac{-1\pm\sqrt{3}i}{2}$$

$$\Rightarrow \quad m=1, \quad m=-\frac{1}{2}\pm\frac{\sqrt{3}}{2}i$$

$$\therefore \quad y_c = c_1e^{x} + \left[c_2\cos\left[\frac{\sqrt{3}}{2}x\right] + c_3\sin\left[\frac{\sqrt{3}}{2}x\right]\right]e^{-x/2}$$

$$\therefore \quad y_p = \frac{3\cos 2x}{f(D)}$$

$$= \frac{3\cos 2x}{D^3-1}$$

$$= \frac{3\cos 2x}{D(D^2)-1}$$

$$= \frac{3\cos 2x}{D(2^2)-1}$$

$$= \frac{3\cos 2x}{-4D-1}$$

$$= -3\,\frac{\cos 2x}{4D+1}$$

$$= -3\,\frac{[4D-1]\cos 2x}{[4D+1][4D-1]}$$

$$= -3\,\frac{(4D-1)\cos 2x}{16D^2-1}$$

$$= -3\,\frac{(4D-1)\cos 2x}{16(-2)^2-1}$$

$$= \frac{-3\left[4D(\cos 2x) - \cos 2x\right]}{-64-1}$$

$$= -\frac{3}{65}\left[-8\sin 2x - \cos 2x\right]$$

$$= -\frac{3}{65}\left[8\sin 2x + \cos 2x\right]$$

$\therefore$ The solution is $y = y_c + y_p$

$$\Rightarrow y = c_1 e^{x} + \left[c_2 \cos\frac{\sqrt{3}}{2}x + c_3 \sin\frac{\sqrt{3}}{2}x\right]e^{-x/2} - \frac{3}{65}\left[8\sin 2x + \cos 2x\right]$$

14. Solve $(D^3 + 2D^2 + D)y = \sin 2x$

$\Rightarrow$ Given :- $f(D)y = \sin 2x$

$\therefore$ The A.E is $f(m) = 0$

$\Rightarrow m^3 + 2m^2 + m = 0$

$\Rightarrow m(m^2 + 2m + 1) = 0$

$\Rightarrow m(m+1)^2 = 0$

$\Rightarrow m = 0, \ (m+1)^2 = 0$

$\Rightarrow m = 0, \ (m+1)(m+1) = 0$

$\Rightarrow m = 0, -1, -1$

$\therefore y_c = c_1 e^{0.x} + (c_2 + c_3 x)e^{-x}$

$\Rightarrow y_c = c_1 + (c_2 + c_3 x)e^{-x}$

$$\therefore \quad y_p = \frac{\sin 2x}{f(D)}$$

$$\Rightarrow \quad y_p = \frac{\sin 2x}{D^3 + 2D^2 + D}$$

$$\Rightarrow \quad y_p = \frac{\sin 2x}{-4D + 2(-4) + D}$$

$$\Rightarrow \quad y_p = \frac{\sin 2x}{-3D - 8}$$

$$\Rightarrow \quad y_p = -\frac{\sin 2x}{3D + 8}$$

$$\Rightarrow \quad y_p = \frac{-(3D-8)\sin 2x}{(3D+8)(3D-8)}$$

$$\Rightarrow \quad y_p = -\frac{(3D-8)\sin 2x}{9D^2 - 64}$$

$$\Rightarrow \quad y_p = \frac{-(3D-8)\sin 2x}{9(-4) - 64}$$

$$\Rightarrow \quad y_p = \frac{-3D(\sin 2x) - 8(\sin 2x)}{-36 - 64}$$

$$\Rightarrow \quad y_p = \frac{1}{100}\left[6\cos 2x - 8\sin 2x\right]$$

$\therefore$ The soln is $y = y_c + y_p$

$$\Rightarrow \quad y = c_1 + (c_2 + c_3 x)e^{-x} + \frac{1}{100}\left[6\cos 2x - 8\sin 2x\right]$$

15. Solve. $\frac{d^2y}{dx^2} - 4y = \cos 2x$

$\Rightarrow$ Given:-

$\Rightarrow (D^2 - 4)y = \cos 2x$

$\Rightarrow f(D)y = \cos 2x$

$\therefore$ The AE is $f(m) = 0$

$\Rightarrow m^2 - 4 = 0$

$\Rightarrow (m-2)(m+2) = 0$

$\Rightarrow m = 2, -2$

$\therefore y_c = c_1 e^{2x} + c_2 e^{-2x}$

$$\therefore y_p = \frac{\cos 2x}{f(D)}$$

$$= \frac{\cos 2x}{D^2 - 4}$$

$$= \frac{\cos 2x}{-4-4}$$

$$= \frac{\cos 2x}{-8}$$

$$= -\frac{1}{8}\cos 2x$$

$\therefore$ The solution is :-

$$y = y_c + y_p$$

$$\Rightarrow y = c_1 e^{2x} + c_2 e^{-2x} - \frac{1}{8}\cos 2x$$

16. Solve. $(D^2-4D+13)y = \cos 2x$

$\Rightarrow$ Given:-

$\Rightarrow f(D)y = \cos 2x$

$\therefore$ The AE is $f(m)=0$

$\Rightarrow m^2-4m+13=0$

$$\Rightarrow m = \frac{-(-4)\pm\sqrt{(-4)^2-4(1)(13)}}{2(1)}$$

$$\Rightarrow m = \frac{4\pm\sqrt{16-52}}{2}$$

$$\Rightarrow m = \frac{4\pm\sqrt{-36}}{2}$$

$$\Rightarrow m = \frac{4\pm 6i}{2}$$

$$\Rightarrow m = 2\pm 3i$$

$$\therefore y_c = (c_1\cos 3x + c_2\sin 3x)e^{2x}$$

$$\therefore y_p = \frac{\cos 2x}{f(D)}$$

$$= \frac{\cos 2x}{D^2-4D+13}$$

$$= \frac{\cos 2x}{-4-4D+13}$$

$$= \frac{\cos 2x}{9-4D}$$

$$= \frac{\cos 2x\,[9+4D]}{[9-4D][9+4D]}$$

$$= \frac{(9+4D)\cos 2x}{81-16D^2}$$

$$= \frac{9\cos 2x + 4D\cos 2x}{81-16(-4)}$$

$$= \frac{9\cos 2x - 8\sin 2x}{145}$$

$\therefore$ The solution is

$$y = y_c + y_p$$

$$\Rightarrow \quad y = (c_1\cos 3x + c_2\sin 3x)e^{2x} + \frac{9\cos 2x - 8\sin 2x}{145}$$

17. Solve $(D^2-4D+4)y = 8(e^{2x} + \sin 2x)$

$\Rightarrow$ Given:-

$$f(D)\,y = 8(e^{2x} + \sin 2x)$$

$\therefore$ The AE is $f(m)=0$

$$\Rightarrow m^2 - 4m + 4 = 0$$

$$\Rightarrow (m-2)^2 = 0$$

$$\Rightarrow m = 2, 2$$

$$\therefore \quad y_c = (c_1 + c_2 x)e^{2x}$$

$$\therefore \quad y_p = \frac{8(e^{2x} + \sin 2x)}{f(D)}$$

$$= \frac{8e^{2x}}{D^2-4D+4} + \frac{8\sin 2x}{D^2-4D+4}$$

$$= \frac{8e^{2x}}{(D-2)^2} + \frac{8\sin 2x}{-2^2-4D+4}$$

$$= \overset{4}{\cancel{8}} \cdot \frac{x^2}{\cancel{2}!} \frac{e^{2x}}{1} + \frac{8\sin 2x}{-4D}$$

$$= 4x^2 e^{2x} - 2D\frac{(\sin 2x)}{D^2}$$

$$= 4x^2 e^{2x} - \frac{2D(\sin 2x)}{-4}$$

$$= 4x^2 e^{2x} + \frac{1}{2}(2\cos 2x)$$

$$\Rightarrow y_p = 4x^2 e^{2x} + \cos 2x$$

$\therefore$ The solution is $y = y_c + y_p$

$$\Rightarrow y = (c_1 + c_2 x)e^{2x} + 4x^2 e^{2x} + \cos 2x$$

18. Solve $y'' + 3y' + 2y = 12x^2$

$\Rightarrow$ Given:-

$$(D^2 + 3D + 2)y = 12x^2$$

$$\Rightarrow f(D)y = 12x^2$$

$\therefore$ The AE is $f(m) = 0$

$$\Rightarrow m^2 + 3m + 2 = 0$$

$$\Rightarrow m^2 + m + 2m + 2 = 0$$

$$\Rightarrow m(m+1) + 2(m+1) = 0$$

$$\Rightarrow (m+2)(m+1) = 0$$

$$\Rightarrow m = -2, -1$$

$$\therefore y_c = c_1 e^{-2x} + c_2 e^{-x}$$

$$\therefore y_p = \frac{12x^2}{f(D)}$$

$$= \frac{12x^2}{D^2+3D+2}$$

$$= \frac{12x^2}{2\left[1+\left(\frac{D^2+3D}{2}\right)\right]}$$

$$= \frac{6x^2}{\left[1+\left[\frac{D^2+3D}{2}\right]\right]}$$

$$= 6\left[1+\left(\frac{D^2+3D}{2}\right)\right]^{-1} x^2$$

$$= 6\left[1-\left(\frac{D^2+3D}{2}\right)+\left(\frac{D^2+3D}{2}\right)^2+ - - -\right] x^2$$

$$= 6\left[1-\frac{1}{2}(D^2+3D)+\frac{1}{4}(D^4+6D^3+9D^2)\right] x^2$$

$$= 6\left[x^2-\frac{1}{2}(D^2+3D)(x^2)+\frac{1}{4}(D^4+6D^3+9D^2)(x^2)\right]$$

$$= 6\left[x^2-\frac{1}{2}(2+6x)+\frac{1}{4}(18)\right]$$

$$= 6\left[x^2-1-3x+\frac{9}{2}\right]$$

$$\Rightarrow y_p = 6\left[x^2-3x+\frac{7}{2}\right]$$

$\therefore$ The solution is, $y = y_c + y_p$

$$y = c_1e^{-2x}+c_2e^{-x}+6\left[x^2-3x+\frac{7}{2}\right]$$

19. Solve $(D^2+D)y = x^2+2x+4$

$\Rightarrow$ Given.

$$f(D)y = x^2+2x+4$$

$\therefore$ The AE is $f(m)=0$

$$\Rightarrow m^2+m=0$$

$$\Rightarrow m(m+1)=0$$

$$\Rightarrow m=0,-1$$

$$\therefore y_c = c_1e^{0.x}+c_2e^{-x}$$

$$\therefore y_P = \frac{x^2+2x+4}{f(D)}$$

$$= \frac{x^2+2x+4}{D^2+D}$$

$$= \frac{\frac{1}{D}(x^2+2x+4)}{1+D}$$

$$= \frac{\int(x^2+2x+4)}{1+D}$$

$$= \frac{\frac{x^3}{3}+x^2+4x}{(1+D)'}$$

$$= (1+D)^{-1}\left[\frac{x^3}{3}+x^2+4x\right]$$

$$= \left[1-D+D^2-D^3+\cdots\right]\left[\frac{x^3}{3}+x^2+4x\right]$$

$$= \left[\frac{x^3}{3}+x^2+4x\right]-D\left[\frac{x^3}{3}+x^2+4x\right]+D^2\left[\frac{x^3}{3}+x^2+4x\right]$$
$$-D\left[\frac{x^3}{3}+x^2+4x\right]$$

$$= \left[\frac{x^3}{3} + x^2 + 4x\right] - \left[x^2 + 2x + 4\right] + (2x+2) - (2)$$

$$\Rightarrow y_p = \frac{x^3}{3} + 4x - 4$$

$\therefore$ The solution is $y = y_c + y_p$

$$\Rightarrow y = c_1 e^{0x} + c_2 e^{-x} + \frac{x^3}{3} + 4x - 4$$

20. Solve $(D^2 + D + 1)y = x^2 + 1$

$\Rightarrow$ Given:-

$$(D^2 + D + 1)y = x^2 + 1$$

$$\Rightarrow f(D)y = x^2 + 1$$

$\therefore$ The AE is $f(m) = 0$

$$\Rightarrow m^2 + m + 1 = 0$$

$$\Rightarrow m = \frac{-1 \pm \sqrt{(1)^2 - 4(1)(1)}}{2(1)}$$

$$\Rightarrow m = \frac{-1 \pm \sqrt{-3}}{2}$$

$$\Rightarrow m = \frac{-1 \pm \sqrt{3}i}{2}$$

$$\Rightarrow m = -\frac{1}{2} \pm \frac{\sqrt{3}i}{2}$$

$$\therefore y_c = \left(c_1 \cos\frac{\sqrt{3}}{2}x + c_2 \sin\frac{\sqrt{3}}{2}x\right) e^{-x/2}$$

$$\therefore y_p = \frac{x^2+1}{f(D)}$$

$$= \frac{x^2+1}{D^2+D+1}$$

$$= \frac{x^2+1}{(1+(D^2+D))}$$

$$= [1+(D^2+D)]^{-1}[x^2+1]$$

$$= [1-(D^2+D)+(D^2+D)^2 - \cdots][x^2+1]$$

$$= [x^2+1]-[D^2+D][x^2+1]+[D^4+2D^3+D^2][x^2+1]$$

$$= [x^2+1]-[2+2x]+[2]$$

$$= x^2+1-\cancel{2}-2x+\cancel{2}$$

$$\Rightarrow y_p = x^2-2x+1$$

$\therefore$ The solution is $y = y_c + y_p$

$$\therefore \quad y = \left(c_1 \cos\frac{\sqrt{3}}{2}x + c_2 \sin\frac{\sqrt{3}}{2}x\right)e^{-x/2} + x^2-2x+1$$

21. Solve $(D-2)^2 y = 8(x^2+\sin 2x + e^{2x})$

$\Rightarrow$ Given:-

$$\Rightarrow f(D)y = 8(x^2+\sin 2x+e^{2x})$$

$\therefore$ The AE is $f(m)=0$

$$\Rightarrow (m-2)^2 = 0$$

$$\Rightarrow m = 2, 2$$

$$\therefore y_c = (c_1+c_2x)e^{2x}$$

$$\therefore y_p = \frac{8(x^2+\sin 2x+e^{2x})}{(D-2)^2}$$

$$= \frac{8x^2}{(D-2)^2} + \frac{8\sin 2x}{D^2-4D+4} + \frac{8e^{2x}}{(D-2)^2}$$

$$= \frac{8}{4} \frac{x^2}{\left(1-\frac{D}{2}\right)^2} + \frac{8\sin 2x}{-4-4D+4} + \frac{8x^2}{2!}\frac{e^{2x}}{1}$$

$$= 2\left(1-\frac{D}{2}\right)^{-2}(x^2) - 2\frac{\sin 2x}{D} + 4x^2 e^{2x}$$

$$= 2\left(1-2\left(\frac{D}{2}\right)+3\left(\frac{D}{2}\right)^2+---\right)x^2 + 2\int \sin 2x\,dx + 4x^2e^{2x}$$

$$= 2\left[1+D+\frac{3}{4}D^2+---\right]x^2 + \frac{\cancel{2}\cos 2x}{2} + 4x^2e^{2x}$$

$$= 2\left[x^2+2x+\frac{3}{4}(2)\right] + \cos 2x + 4x^2e^{2x}$$

$\Rightarrow$ $y_p = 2x^2+4x+3+\cos 2x + 4x^2e^{2x}$

$\therefore$ The solution is $y = y_c + y_p$

$\Rightarrow$ $y = (c_1+c_2)e^{2x} + 2\left[x^2+2x+\frac{3}{4}(2)\right] + \cos 2x + 4x^2e^{2x}$

$\Rightarrow$ $y = (c_1+c_2)e^{2x} + 2x^2+4x+3+\cos 2x + 4x^2e^{2x}$

22. Solve $(D^2+1)y = 3x^2+6x+12$

$\Rightarrow$ Given:-

$f(D)y = 3x^2+6x+12$

$\therefore$ The AE is $f(m)=0$

$\Rightarrow$ $(m^2+1)=0$

$\Rightarrow$ $m^2=-1$

$\Rightarrow$ $m=\pm i$

$\Rightarrow$ $m = 0\pm 1i$

$\therefore$ $y_c = (c_1\cos x + c_2\sin x)e^{0.x}$

$\Rightarrow$ $y_c = c_1\cos x + c_2 \sin x$

$$\therefore \quad y_p = \frac{3x^2+6x+12}{f(D)}$$

$$= \frac{3x^2+6x+12}{D^2+1}$$

$$= (1+D^2)^{-1}(3x^2+6x+12)$$

$$= [1-D^2+D^4-D^6+---][3x^2+6x+12]$$

$$= 3x^2+6x+12-6$$

$$\Rightarrow y_p = 3x^2+6x+6$$

$\therefore$ The solution is :-

$$y = y_c + y_p$$

$$\Rightarrow y = (c_1\cos x + c_2\sin x) + 3x^2+6x+6$$

23. Solve $(D^3+8)y = x^4+2x+1$

$\Rightarrow$ Given :-

$$f(D)\,y = x^4+2x+1$$

$\therefore$ The AE is $f(m)=0$

$$\Rightarrow m^3+8=0$$

$$\Rightarrow (m+2)(m^2-2m+4)=0$$

$$\Rightarrow m+2=0, \quad m = \frac{2 \pm \sqrt{4-4(1)(4)}}{2(1)}$$

$$\Rightarrow m=-2, \quad m = \frac{2 \pm \sqrt{12}}{2}$$

$$\Rightarrow m=-2, \quad m = 1 \pm \sqrt{3}$$

$$\therefore y_c = c_1 e^{-2x} + (c_2\cos\sqrt{3}x + c_3\sin\sqrt{3}x)\,e^{x}$$

$$\therefore y_p = \frac{x^4+2x+1}{f(D)}$$

$$= \frac{x^4+2x+1}{D^3+8}$$

$$= \frac{1}{8} \frac{x^4+2x+1}{\left(1+\frac{D^3}{8}\right)}$$

$$= \frac{1}{8}(x^4+2x+1)\left(1+\frac{D^3}{8}\right)^{-1}$$

$$= \frac{1}{8}(x^4+2x+1)\left(1-\frac{D^3}{8}+\frac{D^6}{64}+---\right)$$

$$= \frac{1}{8}\left[(x^4+2x+1) - \frac{24x}{8}\right]$$

$$= \frac{1}{8}\left[x^4+2x-1-\frac{24x}{8}\right]$$

$$\Rightarrow \quad y_p = \frac{x^4+1-x}{8}$$

$$\therefore \Rightarrow y = y_c + y_p$$

$$y = c_1 e^{-2x} + (c_2 \cos\sqrt{3}x + c_3 \sin\sqrt{3})e^{x} + \frac{x^4+1-x}{8}$$

24. Solve $(D^2+4)y = x^2 + \cos 2x$

$\Rightarrow$ Given:-

$$f(D)y = x^2 + \cos 2x$$

$\therefore$ The AE is $f(m)=0$

$$\Rightarrow m^2+4=0$$

$$\Rightarrow m^2 = -4$$

$$\Rightarrow m^2 = -4$$

$$\Rightarrow m = \pm 2i$$

$$\therefore \quad y_c = c_1 \cos 2x + c_2 \sin 2x$$

$$\therefore \quad y_p = \frac{x^2 + \cos 2x}{f(D)}$$

$$= \frac{x^2 + \cos 2x}{D^2 + 4}$$

$$= \frac{x^2}{D^2+4} + \frac{\cos 2x}{D^2 + 4}$$

$$= \frac{1}{4} \frac{x^2}{\left(1 + \frac{D^2}{4}\right)} + \frac{\cos 2x}{D^2 + 2^2}$$

$$= \frac{1}{4} x^2 \left[1 + \frac{D^2}{4}\right]^{-1} + \frac{x}{4} \sin 2x$$

$$= \frac{1}{4}\left[1 + \frac{D^2}{4} + \frac{D^4}{10} + \cdots\right] x^2 + \frac{x}{4} \sin 2x$$

$$= \frac{1}{4}\left[x^2 + \frac{2}{4} + 0\right] + \frac{x}{4} \sin 2x$$

$$\Rightarrow y_p = \frac{x^2}{4} + \frac{1}{8} + \frac{x}{4} \sin 2x$$

$\therefore$ The solution is $y = y_c + y_p$

$$y = (c_1 \cos 2x + c_2 \sin 2x) + \frac{x^2}{4} + \frac{1}{8} + \frac{x}{4} \sin 2x$$

Linear Differential Equation with Variable Co-efficient.

Let for any $a_0, a_1, a_2 \dots a_n$ and a, b constants, the equation

$$a_0(ax+b)^n \frac{d^n y}{dx^n} + a_1(ax+b)^{n-1}\frac{d^{n-1}y}{dx^{n-1}} + a_2(ax+b)^{n-2}\frac{d^{n-2}y}{dx^{n-2}} \dots a_n y = Q(x) \quad \text{---(1)}$$

is called the Legender's Linear Differential Equation and when $a=1, b=0$, then the equation,

$$(1) \Rightarrow a_0 x^n \frac{d^n y}{dx^n} + a_1 x^{n-1}\frac{d^{n-1}y}{dx^{n-1}} + a_2 x^{n-2}\frac{d^{n-2}y}{dx^{n-2}} + \dots a_n y = Q(x) \quad \text{---(2)}$$

is called the Cauchy's Linear Differential Equation.

Solution of Legender's and Cauchy's D.E

Step 1 :- Write the given D.E to the standard form in the notations of $y, y', y'', y''', \dots$

Step 2 :-

Legender's D.E

1. Let, $\log_e(ax+b) = z$

$\Rightarrow ax+b = e^z$

$\Rightarrow x = \dfrac{e^z - b}{a}$

2. $(ax+b)y' = a.Dy$

$(ax+b)^2 y'' = a^2 . D(D-1)y$

$(ax+b)^3 y''' = a^3 . D(D-1)(D-2)y$

where, $D = \dfrac{d}{dz}$

Cauchy's D.E

1. Let, $\log x = z$

$\Rightarrow x = e^z$

2. $xy' = Dy$

$x^2 y'' = D(D-1)y$

$x^3 y''' = D(D-1)(D-2)y$

where, $D = \dfrac{d}{dz}$

Step 3 :- Write the above in the given D.E and solve

Step 4 :- Finally replace $z = \log(ax+b)$ (or) $z = \log x$ in the solution.

1. Solve $(1+x)^2\frac{d^2y}{dx^2} + (1+x)\frac{dy}{dx} + y = 2\sin(\log(1+x))$

$\Rightarrow$ Given :-

$$(1+x)^2\frac{d^2y}{dx^2} + (1+x)\frac{dy}{dx} + y = 2\sin(\log(1+x)) \quad \text{---(1)}$$

$$\Rightarrow (1+x)^2 y'' + (1+x)y' + y = 2\sin(\log(1+x)) \quad \text{---(2)}$$

Let, $\log_e(1+x) = z$

$\Rightarrow 1+x = e^z$

$\Rightarrow x = e^z - 1$

$\therefore (1+x)y = 1.\,Dy = Dy$

$(1+x)^2 y'' = 1^2.\,D(D-1)y = (D^2-D)y$

where, $D = \frac{d}{dz}$

$(2) \Rightarrow (D^2-D)y + Dy + y = 2\sin z$

$\Rightarrow [D^2 - D + D + 1]y = 2\sin z$

$\Rightarrow [D^2+1]y = 2\sin z$

$\Rightarrow f(D)y = 2\sin z$

$\therefore$ The AE is $f(m) = 0$

$\Rightarrow m^2 + 1 = 0$

$\Rightarrow m = 0 \pm 1i$

$\therefore \quad y_c = c_1 \cos z + c_2 \sin z$

$$\therefore \quad y_p = \frac{2\sin z}{f(D)}$$

$$= \frac{2\sin z}{D^2+1}$$

$$= -2 \cdot \frac{z}{2(1)} \cos z$$

$\Rightarrow y_p = -z\cos z$

$\therefore$ The solution is $y = y_c + y_p$

$\Rightarrow \quad y = c_1 \cos z + c_2 \sin z - z\cos z$

$$\Rightarrow \quad y = c_1 \cos[\log(1+x)] + c_2 \sin[\log(1+x)] - \log(1+x) \cos[\log(1+x)]$$

2. Solve $(1+x^2)\frac{d^2y}{dx^2} + (1+x)\frac{dy}{dx} + y = \sin 2[\log(1+x)]$

$\Rightarrow$ Given:-

$$(1+x^2)\frac{d^2y}{dx^2} + (1+x)\frac{dy}{dx} + y = \sin[2\log(1+x)] \quad \text{——(1)}$$

$$\Rightarrow (1+x^2)y'' + (1+x)y' + y = \sin[2\log(1+x)] \quad \text{——(2)}$$

Let, $\log_e(1+x) = z$

$\Rightarrow (1+x) = e^z$

$\Rightarrow \quad x = e^z - 1$

$\therefore (1+x)y' = 1. Dy = Dy$

$(1+x)^2 y'' = 1^2. D(D-1)y = (D^2-D)y$

where, $D = \frac{d}{dz}$

$\therefore$ (2) $\Rightarrow (D^2-D)y + Dy + y = \sin 2z$

$\Rightarrow [D^2 - D + D + 1]y = \sin 2z$

$\Rightarrow [D^2+1]\, y = \sin 2z$

$\Rightarrow f(D)y = \sin 2z$

$\therefore$ The AE is $f(m) = 0$

$\Rightarrow m^2+1 = 0$

$\Rightarrow m^2 = -1$

$\Rightarrow m = 0 \pm 1i$

$\therefore y_c = C_1 \cos z + C_2 \sin z$

$$\therefore y_p = \frac{\sin 2z}{f(D)}$$

$$= \frac{\sin 2z}{D^2+1}$$

$$= \frac{\sin 2x}{-4+1}$$

$$\Rightarrow y_p = -\frac{1}{3} \sin 2z$$

$\therefore$ The solution is, $y = y_c + y_p$

$\Rightarrow y = C_1 \cos z + C_2 \sin z - \frac{1}{3} \sin 2z$

$\Rightarrow y = C_1 \cos[\log(1+x)] + C_2 \sin[\log(1+x)] - \frac{1}{3} \sin 2[\log(1+x)]$

3. Solve $(2x+3)^2 y'' + (2x+3)y' - 12y = 6x$

$\Rightarrow$ Given:-

$$(2x+3)^2 y'' + (2x+3)y' - 12y = 6x \quad \text{——— (1)}$$

Let, $\log_e(2x+3) = z$

$$(2x+3) = e^z$$

$$2x = e^z - 3$$

$$x = \frac{e^z - 3}{2}$$

$$\therefore \; (2x+3)\,y' = 2Dy$$

$$(2x+3)^2 y'' = 2^2 D(D-1)y = 4(D^2 - D)y$$

where, $D = \dfrac{d}{dz}$

$$(1) \Rightarrow \; 4(D^2 - D)y + 2Dy - 12y = 6\left[\frac{e^z - 3}{2}\right]$$

$$\Rightarrow (4D^2 - 4D + 2D - 12)\,y = 3[e^z - 3]$$

$$\Rightarrow (4D^2 - 2D - 12)\,y = 3[e^z - 3]$$

$$\Rightarrow (2D^2 - D - 6)\,y = \frac{3}{2}[e^z - 3]$$

$$\Rightarrow f(D)\,y = \frac{3}{2}[e^z - 3]$$

$\therefore$ The AE is $f(m) = 0$

$$\Rightarrow 2m^2 - m - 6 = 0$$

$$\Rightarrow 2m^2 - 4m + 3m - 6 = 0$$

$$\Rightarrow 2m(m-2) + 3(m-2) = 0$$

$$\Rightarrow (2m+3)(m-2) = 0$$

$$\Rightarrow 2m + 3 = 0, \quad m - 2 = 0$$

$$\Rightarrow m = -\frac{3}{2},\ 2$$

$$\therefore\ y_c = C_1 e^{-3/2\,z} + C_2 e^{2z}$$

$$\therefore\ y_p = \frac{\frac{3}{2}\left(e^z - 3\right)}{f(D)}$$

$$= \frac{3}{2}\left[\frac{e^z}{2D^2 - D - 6} - \frac{3e^{0.z}}{2D^2 - D - 6}\right]$$

$$= \frac{3}{2}\left[\frac{e^z}{2(1) - 1 - 6} - \frac{3}{2(0) - 0 - 6}\right]$$

$$= \frac{3}{2}\left[\frac{e^z}{-5} - \frac{3}{(-6)}\right]$$

$$\Rightarrow y_p = -\frac{3}{10} e^z + \frac{3}{4}$$

$$\therefore\ y = y_c + y_p$$

$$= C_1 e^{-\frac{3}{2}z} + C_2 e^{2z} - \frac{3}{10} e^z + \frac{3}{4}$$

$$= \frac{C_1}{(e^z)^{3/2}} + C_2 (e^z)^2 - \frac{3}{10} e^z + \frac{3}{4}$$

$$= \frac{C_1}{(2x+3)^{3/2}} + C_2 (2x+3)^2 - \frac{3}{10}(2x+3) + \frac{3}{4}$$

4. Solve $(x^2)y'' - xy' + y = \log x$

$\Rightarrow$ Given:-

$$x^2 y'' - xy' + y = \log x \quad \text{---(1)}$$

Let, $\log x = z$

$$\Rightarrow x = e^z$$

Let, $xy' = Dy$

$x^2y'' = D(D-1)y$

where, $D = \frac{d}{dz}$

$\therefore (1) \Rightarrow D(D-1)y - Dy + y = z$

$\Rightarrow (D^2 - D - D + 1)y = z$

$\Rightarrow (D^2 - 2D + 1)y = z$

$\Rightarrow (D^2 - 1)y = z$

$\Rightarrow f(D)y = z$

$\therefore$ The AE is $f(m) = 0$

$\Rightarrow (m-1)^2 = 0$

$\Rightarrow m = 1, 1$

$\therefore y_c = (c_1 + c_2 z)e^z$

$\therefore y_p = \frac{z}{f(D)}$

$\Rightarrow y_p = \frac{z}{(D-1)^2}$

$\Rightarrow y_p = \frac{z}{(1-D)^2}$

$\Rightarrow y_p = (1-D)^{-2} z$

$\Rightarrow y_p = (1 + 2D + 3D^2 + \cdots)z$

$\Rightarrow y_p = z + 2D(z) + 3D^2(z)$

$\Rightarrow y_p = z + 2(1) + D$

$\Rightarrow \quad y_p = z + 2$

$\therefore$ The solution is $y = y_c + y_p$

$\therefore \quad y = (c_1 + c_2 z) e^z + z + 2$

$= (c_1 + c_2 \log x) x + \log x + 2$

5. Solve $x^2 y'' - 3xy' + 4y = (1+x)^2$

$\Rightarrow$ Given :-

$$x^2 y'' - 3xy' + 4y = (1+x)^2 \quad \text{——(1)}$$

Let, $\log x = z$

$\Rightarrow \quad x = e^z$

and $xy' = Dy$

$x^2 y'' = D(D-1)y$

where, $D = \frac{d}{dz}$

$(1) \Rightarrow (D^2 - D)y - 3Dy + 4y = (1+e^z)^2$

$\Rightarrow \quad (D^2 - D - 3D + 4)y = e^{2z} + 2e^z + 1$

$\Rightarrow \quad (D^2 - 4D + 4)y = e^{2z} + 2e^z + 1$

$\Rightarrow \quad (D-2)^2 y = e^{2z} + 2e^z + 1$

$\Rightarrow \quad f(D)y = e^{2z} + 2e^z + 1$

$\therefore$ The AE is $f(m) = 0$

$\Rightarrow \quad (m-2)^2 = 0$

$\Rightarrow \quad m = 2, 2$

$\therefore \quad y_c = (c_1 + c_2 x) e^{2x}$

$$\therefore\ y_p = \frac{e^{2z}+2e^{z}+1}{f(D)}$$

$$= \frac{e^{2z}}{(D-2)^2} + \frac{2e^{z}}{(D-2)^2} + \frac{1.e^{0.z}}{(D-2)^2}$$

$$= \frac{z^2}{2!}\frac{e^{2z}}{1} + \frac{2e^{z}}{(1-2)^2} + \frac{1.e^{0.z}}{(0-2)^2}$$

$$y_p = \frac{z^2}{2}e^{2z} + 2e^{z} + \frac{1}{4}$$

$\therefore$ The solution is, $y = y_c + y_p$

$$y = (c_1+c_2 z)e^{2z} + \frac{z^2}{2}e^{2z} + 2e^{z} + \frac{1}{4}$$

$$y = (c_1 + c_2 \log x)x^2 + \frac{1}{2}(\log x)^2 x^2 + 2x + \frac{1}{4}$$

6. Solve $x\frac{d^2y}{dx^2} - 2\frac{dy}{dx} = x + \frac{1}{x^2}$

$\Rightarrow$ Given:-

$$\Rightarrow x^2y'' - 2xy' = x^2 + \frac{1}{x} \quad \text{—— (1)}$$

Let, $\log x = z$

$x = e^{z}$

$\Rightarrow xy = Dy$

$\Rightarrow x^2y'' = D(D-1)y$

where, $D = \frac{d}{dz}$

$$(1) \Rightarrow D(D-1)y - 2Dy = (e^{z})^2 + \frac{1}{e^{z}}$$

$$\Rightarrow (D^2 - D - 2D)y = e^{2z} + e^{-z}$$

$$\Rightarrow (D^2 - 3D)y = e^{2z} + e^{-z}$$

$\Rightarrow \quad f(D)y = e^{2z} + e^{-z}$

$\therefore$ The AE is $f(m) = 0$

$\Rightarrow \quad m^2 - 3m = 0$

$\Rightarrow \quad m(m-3) = 0$

$\Rightarrow \quad m = 0, 3$

$\therefore \quad y_c = C_1 + C_2 e^{3z}$

$$\therefore \quad y_p = \frac{e^{2z} + e^{-z}}{f(D)}$$

$$= \frac{e^{2z}}{D^2 - 3D} + \frac{e^{-z}}{D^2 - 3D}$$

$$= \frac{e^{2z}}{(2)^2 - 3(2)} + \frac{e^{-z}}{(-1)^2 - 3(-1)}$$

$$= \frac{e^{2z}}{-2} + \frac{e^{-z}}{4}$$

$\therefore$ The solution is $y = y_c + y_p$

$$y = C_1 + C_2 e^{3z} - \frac{1}{2}e^{2z} + \frac{1}{4}e^{-z}$$

$$= C_1 + C_2 (e^z)^3 - \frac{1}{2}(e^z)^2 + \frac{1}{4} \cdot \frac{1}{e^z}$$

$$= C_1 + C_2 x^3 - \frac{1}{2}x^2 + \frac{1}{4x}$$

Method of Variation of Parameters :-

Step 1 :- let the 2nd order D.E with constant co-effients be,

$$a_0 \frac{d^2y}{dx^2} + a_1 \frac{dy}{dx} + a_2 y = Q(x) \quad \text{——(1)}$$

where, a_0, a_1, a_2 are the constants and 'Q' is a function of 'x'.

Step 2 :- Write the given D.E as $f(D)y = Q(x)$ and obtain the complementary function

$$y_c = c_1 y_1 + c_2 y_2$$

Step 3 :- Write the complete solution by replacing c_1 and c_2 as A and B and write

$$y = A(x) y_1 + B(x) y_2$$

where, $A = -\int \frac{y_1 Q(x)}{\omega} dx + K_1$

$$B = \int \frac{y_2 Q(x)}{\omega} dx + K_2$$

$$\omega = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = y_1 y_2' - y_2 y_1'$$

1. Solve $\frac{d^2y}{dx^2} + y = \sec x \cdot \tan x$

$\Rightarrow$ Given :-

$\Rightarrow (D^2+1)\,y = \sec x \cdot \tan x$

$\Rightarrow f(D)y = Q(x)$

$\therefore$ The AE is $f(m)=0$

$\Rightarrow m^2+1=0$

$\Rightarrow m^2=-1$

$\Rightarrow m=\pm i$

$\Rightarrow m=0\pm 1i$

$\therefore y_c = (c_1\cos x + c_2\sin x)e^{0.x}$

$\Rightarrow y_c = c_1\cos x + c_2\sin x$

$\Rightarrow y_c = c_1y_1 + c_2y_2$

$\therefore y_1 = \cos x \qquad y_1' = -\sin x$

$y_2 = \sin x \qquad y_2' = \cos x$

$W = y_1y_2' - y_2y_1'$

$= \cos x \cdot \cos x - \sin x(-\sin x)$

$= \cos^2 x + \sin^2 x$

$W = 1$

$\therefore$ The solution is

$y = Ay_1 + By_2$

$$\therefore A = -\int \frac{y_1 Q(x)}{\omega}\, dx + K_1$$

$$= -\int \frac{\cancel{\cos x} \cdot \cancel{\sec x} \cdot \tan x}{1}\, dx + K_1$$

$$= -\int \tan x \, dx + K_1$$

$$= -\log|\sec x| + K_1$$

$$= \log\left|\frac{1}{\sec x}\right| + K_1$$

$$\Rightarrow A = \log|\cos x| + K_1$$

$$\therefore B = \int \frac{y_2 Q(x)}{\omega}\, dx + K_2$$

$$= \int \frac{\sin x \cdot \sec x \cdot \tan x}{1}\, dx + K_2$$

$$= \int \frac{\sin x}{\cos x} \cdot \tan x \, dx + K_2$$

$$= \int \tan^2 x \, dx + K_2$$

$$= \int (\sec^2 x - 1) + K_2$$

$$\Rightarrow B = \operatorname{Tan} x - x + K_2$$

$$\therefore y = [\log|\cos x| + K_1 + \operatorname{Tan} x - x + K_2]$$

2. Solve $\frac{d^2y}{dx^2} + y = \tan x$

$\Rightarrow$ Given :-

$$(D^2+1)\,y = \tan x$$

$$\Rightarrow f(D)\,y = \tan x$$

$\therefore$ The AE is $f(m)=0$

$$\Rightarrow (m^2+1)=0$$

$$\Rightarrow m^2 = -1$$

$$\Rightarrow m = 0 + 1i$$

$$\therefore y_c = (c_1\cos x + c_2\sin x)\,e^{0.x}$$

$$\Rightarrow y_c = c_1\cos x + c_2\sin x$$

$$\Rightarrow y_c = c_1 y_1 + c_2 y_2$$

$\therefore\ y_1 = \cos x \qquad y_1' = -\sin x$

$y_2 = \sin x \qquad y_2' = \cos x$

$$\therefore W = y_1 y_2' - y_2 y_1'$$

$$= \cos x \cdot \cos x - \sin x(-\sin x)$$

$$= \cos^2 x + \sin^2 x$$

$$\Rightarrow W = 1$$

$\therefore$ The solution is

$$y = A y_1 + B y_2$$

$$A = -\int \frac{y_1\, Q(x)}{\omega}\,dx + K_1$$

$$= -\int \frac{\cos x \cdot \tan x}{1}\,dx + K_1$$

$$= -\int \sin x . dx + K_1$$

$$= -(-\cos x) + K_1$$

$$\Rightarrow A = \cos x + K_1$$

$$\therefore \quad B = \int \frac{y_2 Q(x)}{\omega} dx + K_2$$

$$= \int \frac{\sin x . \tan x}{1} dx + K_2$$

$$= \int \frac{\sin^2 x}{\cos x} dx + K_2$$

$$= \int \frac{1 - \cos^2 x}{\cos x} dx + K_2$$

$$= \int \frac{1}{\cos x} - \frac{\cos^2 x}{\cos x} dx + K_2$$

$$= \int \sec x \, dx - \int \cos x \, dx + K_2$$

$$\Rightarrow B = \log|\sec x + \tan x| - \sin x + K_2$$

$$\therefore \; y = [\cos x + K_1 (\cos x) + \log|\sec x + \tan x| (-\sin x) + K_2]$$

$$y = (\cos x + K_1)(\cos x) - (\log|\sec x + \tan x| (\sin x) + K_2)$$

3. Solve $\frac{d^2y}{dx^2} + y = \sec x$

$\Rightarrow$ Given :- $(D^2+1)y = \sec x$

$= f(D)y = \sec x$

$\Rightarrow f(D)y = Q(x)$

$\therefore$ The AE is $f(m)=0$

$\Rightarrow m^2+1=0$

$\Rightarrow m^2 = -1$

$\Rightarrow m = 0 \pm 1i$

$\therefore y = c_1 \cos x + c_2 \sin x$

$\Rightarrow y = c_1 y_1 + c_2 y_2$

$\therefore y_1 = \cos x \qquad y_1' = -\sin x$

$y_2 = \sin x \qquad y_2' = \cos x$

$W = y_1 y_2' - y_2 y_1'$

$= \cos x \cdot \cos x - \sin x(-\sin x)$

$= \cos^2 x + \sin^2 x$

$= 1$

$\therefore$ The solution is $y = Ay_1 + By_2$

$$\therefore A = -\int \frac{y_1 Q(x)}{W} dx + K_1$$

$$= -\int \frac{\cos x \cdot \sec x}{1} dx + K_1$$

$$= -\int \cancel{\cos x} \cdot \frac{1}{\cancel{\cos x}} dx + K_1$$

$$= -\int 1\, dx + K_1 = -x + K_1$$

$$\therefore \quad B = \int \frac{y_2 Q(x)}{W}\, dx + K_2$$

$$= \int \frac{\sin x \cdot \sec x}{1}\, dx + K_2$$

$$= \int \frac{\sin x}{\cos x}\, dx + K_1$$

$$= \int \tan x + K_2$$

$$\Rightarrow \quad B = \log|\sec x| + K_2$$

$$\therefore \quad y = Ay_1 + By_2$$

$$\Rightarrow y = (-x + K_1)(\cos x) + (\log|\sec x| + K_2)(\sin x)$$

4. Solve by the method of variation of parameter $y'' - 6y' + 9y = \frac{e^{3x}}{x^2}$

$\Rightarrow$ Given:-

$$(D^2 + 6D + 9)\, y = \frac{e^{3x}}{x^2}$$

$$\Rightarrow \quad f(D)\, y = Q(x)$$

$\therefore$ The AE is $f(m) = 0$

$$\Rightarrow \quad m^2 - 6m + 9 = 0$$

$$\Rightarrow \quad m^2 - 3m - 3m + 9 = 0$$

$$\Rightarrow \quad m(m-3) - 3(m-3) = 0$$

$$\Rightarrow \quad (m-3)(m-3) = 0$$

$$\Rightarrow \quad m = 3, 3$$

$$\therefore \quad y_c = (c_1 + c_2 x)e^{3x}$$

$$\Rightarrow \quad y_c = c_1 e^{3x} + c_2 e^{3x}$$

$$\Rightarrow \quad y_c = c_1 y_1 + c_2 y_2$$

$$\therefore \quad y_1 = e^{3x} \qquad y_1' = 3e^{3x}$$

$$y_2 = xe^{3x} \qquad y_2' = e^{3x} + 3xe^{3x}$$

$$\therefore \quad W = y_1 y_2' - y_2 y_1'$$

$$= e^{3x}\left[e^{3x} + 3e^{3x}\right] - xe^{3x}\left[3e^{3x}\right]$$

$$= e^{6x} + \cancel{3xe^{6x}} - \cancel{3xe^{6x}}$$

$$\Rightarrow \quad W = e^{6x}$$

$$\therefore \quad A = -\int \frac{y_1 Q(x)}{W}\, dx + K_1$$

$$= -\int \frac{e^{3x} \cdot \frac{e^{3x}}{x^2}}{e^{6x}}\, dx + K_1$$

$$= -\int \frac{\frac{\cancel{e^{6x}}}{x^2}}{\cancel{e^{6x}}}\, dx + K_1$$

$$= -\int \frac{1}{x^2}\, dx + K_1$$

$$= -\left[-\frac{1}{x}\right] + K_1$$

$$\Rightarrow \quad A = \frac{1}{x} + K_1$$

$$\therefore \quad B = \int \frac{y_2 Q(x)}{W}\, dx + K_2$$

$$= \int \frac{x\cancel{e^{3x}} \cdot \frac{e^{3x}}{x^2}}{e^{6x}}\, dx + K_2$$

$$= \int \frac{1}{x}\, dx + K_2$$

$$\Rightarrow B = \log x + K_2$$

$$\therefore \quad y = Ay_1 + By_2$$

$$= \left(\frac{1}{x} + K_1\right)\left[e^{3x}\right] + \left(\log x + K_2\right)\left[xe^{3x}\right]$$

5. Solve $\frac{d^2y}{dx^2} - y = \frac{2}{1+e^x}$

$\Rightarrow$ Given :-

$$(D^2 - 1)y = \frac{2}{1+e^x}$$

$$\Rightarrow f(D)y = Q(x)$$

$\therefore$ The AE is $f(m) = 0$

$$\Rightarrow m^2 - 1 = 0$$

$$\Rightarrow m^2 = 1$$

$$\Rightarrow m = \pm 1$$

$$\Rightarrow m = -1, 1$$

$$\therefore y_c = c_1 e^{-x} + c_2 e^x$$

$$y_c = c_1 y_1 + c_2 y_2$$

$\therefore \quad y_1 = e^{-x} \qquad y_1' = -e^{-x}$

$\quad\;\; y_2 = e^x \qquad y_2' = e^x$

$$\therefore A = -\int \frac{y_1 Q(x)}{\omega}\, dx + K_1$$

$$= -\int \frac{e^x \cdot \frac{2}{1+e^x}}{2}\, dx + K_1$$

$$= -\int \frac{e^x}{1+e^x}\, dx + K_1$$

$$= -\int \frac{1}{e^x(1+e^x)}\, dx + K_1$$

$$= -\int \frac{e^x}{(e^x)^2(1+e^x)}\, dx + K_1$$

Let, $e^x = t$

$\Rightarrow e^x dx = dt$

$$\therefore A = -\int \frac{1}{t^2(1+t)}\, dt + K_1$$

$$\frac{1}{t^2(1+t)} = \frac{p}{t} + \frac{q}{t^2} + \frac{r}{1+t}$$

$$\Rightarrow 1 = pt(1+t) + q(1+t) + rt^2 \quad \text{—— (1)}$$

where, $t=0$

(1) $\Rightarrow$ $1 = q$

$\Rightarrow$ $q = 1$ when $t = -1$

(1) $\Rightarrow$ $1 = r(-1)^2 \Rightarrow r = 1$

$p + r = 0$

$\Rightarrow$ $p = -r$

$\Rightarrow$ $p = -1$

$$\therefore \frac{1}{t^2(1+t)} = -\frac{1}{t} + \frac{1}{t^2} + \frac{1}{1+t}$$

$$\therefore A = -\int\left[-\frac{1}{t} + \frac{1}{t^2} + \frac{1}{1+t}\right] dt + K_1$$

$$\Rightarrow A = \int \frac{1}{t}\, dt - \int \frac{1}{t^2}\, dt \int \frac{1}{1+t}\, dt + K_1$$

$$\Rightarrow A = \log t - \frac{1}{t} - \log(1+t) + K_1$$

$$\Rightarrow A = \log(e^x) + \frac{1}{e^x} \log(1+e^x) + K_1$$

$$\Rightarrow A = x + e^{-x} - \log(1+e^x) + K_1$$

$$\therefore B = \int \frac{y_2 Q(x)}{W}\, dx + K_2$$

$$= \int \frac{e^x \cdot \frac{x}{1+e^x}}{x}\, dx + K_2$$

$$= \int \frac{e^x}{1+e^x}\, dx + K_2$$

$$\Rightarrow B = \log(1+e^x) + K_2$$

$\therefore$ The solution is

$$\Rightarrow y = \left[x + e^{-x} - \log(1+e^x) + K_1\right] e^{-x} + \left[\log(1+e^x) + K_2\right] e^x$$

6. Solve $\dfrac{d^2y}{dx^2} - 2\dfrac{dy}{dx} = e^x \sin x$

$\Rightarrow$ Given:- $(D^2 - 2D)y = e^x \sin x$

$\Rightarrow \quad f(D)y = Q(x)$

$\therefore$ The AE is $f(m)=0$

$\Rightarrow \quad (m^2 - 2m) = 0$

$\Rightarrow \quad m(m-2) = 0$

$\Rightarrow \quad m = 0, \; m = 2$

$\therefore \quad y_c = c_1 \cdot e^{0 \cdot x} + c_2 e^{2x}$

$\Rightarrow \quad y_c = c_1 + c_2 e^{2x}$

$\Rightarrow \quad y_c = c_1 y_1 + c_2 y_2$

$\therefore \quad y_1 = 1 \qquad y_1' = 0$

$y_2 = e^{2x} \qquad y_2' = 2e^{2x}$

$\therefore \quad W = y_1 y_2' - y_2 y_1'$

$= (1)(2e^{2x}) - e^{2x}(0)$

$= 2e^{2x}$

$$\therefore A = -\int \frac{y_1 Q(x)}{W}\,dx + k_1$$

$$= -\int \frac{1 \cdot e^x \sin x}{2e^{2x}}\,dx + k_1$$

$$= -\frac{1}{2}\int \frac{\sin x}{e^x}\,dx + k_1$$

$$= -\frac{1}{2}\int e^{-x}\sin x\,dx + k_1$$

$$= -\frac{1}{2}\left[\frac{e^{-x}}{(-1)^2+1^2}(-\sin x-\cos x)\right] + K_1$$

$$\Rightarrow A = \frac{1}{4}e^{-x}[\sin x + \cos x] + K_1$$

$$\therefore B = \int \frac{y_2 Q(x)}{W}\,dx + K_2$$

$$= \int \frac{e^{2x}\cdot e^{x}\sin x}{2e^{2x}}\,dx + K_2$$

$$= \frac{1}{2}\int e^{x}\sin x\,dx + K_2$$

$$= \frac{1}{2}\frac{e^{x}}{2}[\sin x - \cos x] + K_2$$

$$\Rightarrow B = \frac{e^{x}}{4}[\sin x - \cos x] + K_2$$

$\therefore$ The solution is $y = Ay_1 + By_2$

$$\Rightarrow y = \left(\frac{1}{4}e^{-x}[\sin x + \cos x] + K_1\right) + \left(\frac{e^{x}}{4}[\sin x - \cos x] + K_2\right)(e^{2x})$$

7. Solve $(D^2+4)y = \tan 2x$

$\Rightarrow$ Given:-

$$(D^2+4)y = \tan 2x$$

$$\Rightarrow f(D)\,y = \tan 2x$$

$\therefore$ The AE is $f(m)=0$

$$f \Rightarrow (m^2+4) = 0$$

$$\Rightarrow m^2 = -4$$

$$\Rightarrow m = 0 \pm 2i$$

$$\therefore \quad y_c = c_1\cos 2x + c_2\sin 2x$$

$$\therefore \quad y_c = c_1y_1 + c_2y_2$$

$$\therefore \quad y_1 = \cos 2x \Rightarrow y_1' = -2\sin 2x$$

$$y_2 = \sin 2x \Rightarrow y_2' = 2\cos 2x$$

$$\therefore \quad W = y_1y_2' - y_2y_1'$$

$$= (\cos 2x)(2\cos 2x) - (\sin 2x)(-2\sin 2x)$$

$$= 2(\cos^2 2x + \sin^2 2x)$$

$$\Rightarrow W = 2$$

$$\therefore \quad A = -\int \frac{y_1 Q(x)}{W}\,dx + K_1$$

$$= -\int \frac{(\cos 2x)(\tan 2x)}{2}\,dx + K_1$$

$$= -\frac{1}{2}\int \sin 2x\,dx + K_1$$

$$= -\frac{1}{2}\left[-\frac{\cos 2x}{2}\right] + K_1$$

$$= \frac{1}{4}\cos 2x + K_1$$

$$\therefore \quad B = \int \frac{y_2 Q(x)}{W}\,dx + K_2$$

$$= \int \frac{(\sin 2x)(\tan 2x)}{2}\,dx + K_2$$

$$= \frac{1}{2}\int \frac{\sin^2 2x}{\cos 2x}\,dx + K_2$$

$$= \frac{1}{2}\int \frac{1-\cos^2 2x}{\cos 2x}\,dx + K_2$$

$$= \frac{1}{2}\int \frac{1-\cos^2 2x}{\cos 2x}\,dx + K_2$$

$$= \frac{1}{2}\int (\sec 2x - \cos 2x)\,dx + K_2$$

$$= \frac{1}{2}\left[\frac{\log|\sec 2x + \tan 2x|}{2} - \frac{1}{2}\sin 2x\right] + K_2$$

$$\Rightarrow B = \frac{1}{4}\left[\log|\sec 2x + \tan 2x| - \sin 2x\right] + K_2$$

$\therefore$ The solution is $y = Ay_1 + By_2$

$$\Rightarrow y = \left[\frac{1}{4}\cos 2x + K_1\right](\cos 2x) + \left[\frac{1}{4}\left[\log|\sec 2x + \tan 2x| - \sin 2x\right] + K_2\right](\sin 2x)$$

THANK YOU