Module-3

FOURIER TRANSFORMS

Definition :-

① Let $F(x)$ is a function defined on $[-\infty, \infty]$, then the Fourier transform can be defined as

$$F[F(x)] = \int_{-\infty}^{\infty} e^{isx} F(x)\,dx = f(s)$$

where 'F' is called the Fourier transform operator and 's' be the parameter either real (or) complex.

The inverse Fourier transform is defined as

$$F^{-1}[f(s)] = F(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-isx} f(s)\,ds$$

② Let $F(x)$ be a function defined on $[0, \infty]$, then the Fourier cosine transform of $F(x)$ is defined as

$$F_c[F(x)] = \int_0^{\infty} F(x)\cos(sx)\,dx = f_c(s).$$

and its inverse fourier cosine transform is

$$F^{-1}[f_c(s)] = F(x) = \frac{2}{\pi}\int_0^{\infty} f_c(s)\cos(sx)\,ds$$

Similarly, the fourier sin transform of $F(x)$ is $F_s[F(x)] = \int_0^{\infty} F(x)\sin(sx)\,dx = f_s(s)$ and its inverse sin transform is $F^{-1}[f_s(s)] = F(x) = \frac{2}{\pi}\int_0^{\infty} f_s(s)\sin(sx)\,ds.$

① Find the fourier transform of $e^{-a^2x^2}$, $a>0$ and hence show that the fourier transform of $e^{-x^2/2}$ is $\sqrt{2\pi}\,e^{-s^2/2}$

Given $F(x) = e^{-a^2x^2}$, $a>0$

WKT

$$F[F(x)] = \int_{-\infty}^{\infty} e^{isx} F(x)\,dx$$

$$= \int_{-\infty}^{\infty} e^{-a^2x^2} e^{isx}\,dx$$

$$= \int_{-\infty}^{\infty} e^{-a^2x^2+isx}\,dx$$

$$= \int_{-\infty}^{\infty} e^{-(a^2x^2-isx)}\,dx$$

$$= \int_{-\infty}^{\infty} e^{-\left[(ax)^2 - 2(ax)\left(\frac{is}{2a}\right) + \left(\frac{is}{2a}\right)^2 - \left(\frac{is}{2a}\right)^2\right]}\,dx$$

$$= \int_{-\infty}^{\infty} e^{-\left\{\left(ax-\frac{is}{2a}\right)^2 - \frac{i^2s^2}{4a^2}\right\}}\,dx$$

$$= \int_{-\infty}^{\infty} e^{-\left\{\left(ax-\frac{is}{2a}\right)^2 + \frac{s^2}{4a^2}\right\}}\,dx$$

$$= \int_{-\infty}^{\infty} e^{-\left(ax-\frac{is}{2a}\right)^2} e^{-s^2/4a^2}\,dx$$

$$= e^{-s^2/4a^2} \int_{-\infty}^{\infty} e^{-\left(ax-\frac{is}{2a}\right)^2}\,dx$$

Let $ax - \frac{is}{2a} = u$

d w.r.t 'x'

$a\,dx = du$

$dx = \frac{1}{a}\,du$

when $x = \infty \Rightarrow u = \infty$

$x = -\infty \Rightarrow u = -\infty$

$$F[F(x)] = e^{-s^2/4a^2} \int_{-\infty}^{\infty} e^{-u^2} \cdot \frac{1}{a}\,du$$

$$= \frac{1}{a} e^{-s^2/4a^2} \int_{-\infty}^{\infty} e^{-u^2}\,du$$

$$= \frac{1}{a} e^{-s^2/4a^2} \sqrt{\pi}$$

$$F[e^{-a^2x^2}] = \frac{\sqrt{\pi}}{a} e^{-s^2/4a^2} \quad -①$$

Let $a = \frac{1}{\sqrt{2}}$

$$① \Rightarrow F[e^{-x^2/2}] = \frac{\sqrt{\pi}}{\frac{1}{\sqrt{2}}} \cdot e^{-s^2/4\left(\frac{1}{2}\right)}$$

$$F[e^{-x^2/2}] = \sqrt{2\pi} e^{-s^2/2}$$

② Find the Fourier transform of $F(x) = \begin{cases} 1, & |x| < a \\ 0, & |x| \geq a \end{cases}$ and hence find $\int_0^\infty \frac{\sin x}{x} dx$

Given $F(x) = \begin{cases} 1, & |x| < a \\ 0, & |x| \geq a \end{cases}$

$$F(x) = \begin{cases} 1, & -a < x < a \\ 0, & x \geq a \end{cases}$$

WKT $F[F(x)] = \int_{-\infty}^{\infty} F(x) e^{isx} dx$

$$= \int_{-\infty}^{-a} F(x) e^{isx} dx + \int_{-a}^{a} F(x) e^{isx} dx + \int_{a}^{\infty} F(x) e^{isx} dx$$

$$= 0 + \int_{-a}^{a} 1 e^{isx} dx + 0$$

$$F[F(x)] = \int_{-a}^{a} e^{isx} dx$$

$$= \left[\frac{e^{isx}}{is}\right]_{-a}^{a}$$

$$= \frac{1}{is}\left[e^{ias} - e^{-ias}\right]$$

$$= \frac{1}{is}\left[\cos(as) + i\sin(as) - \cos(as) + i\sin(as)\right]$$

$$= \frac{1}{\cancel{i}s}\left[2\cancel{i}\sin(as)\right]$$

$$f(s) = \frac{2}{s} \sin as$$

WKT

$$F^{-1}[f(s)] = \frac{1}{2\pi}\int_{-\infty}^{\infty} f(s)\, e^{-isx}\, dx = F(x)$$

$$= \int_{-\infty}^{\infty} \frac{2}{s}\sin(as)\, e^{-isx}\, dx = 2\pi\, F(x) = 2\pi \begin{cases} 1, & a < x < a \\ 0 & x \geq a \end{cases}$$

Let $x = 0$

$$① \Rightarrow \int_{-\infty}^{\infty} \frac{2}{s} \sin(as)\, ds = 2\pi(1)$$

$$\Rightarrow \int_{-\infty}^{\infty} \frac{\cancel{2}}{s} \sin(as)\, ds = \cancel{2}\pi(1)$$

$$\Rightarrow 2\int_{-\infty}^{\infty} \frac{\sin as}{s}\, ds = \pi$$

$$= \int_{0}^{\infty} \frac{\sin as}{s}\, ds = \frac{\pi}{2} \quad - ②$$

when $a = 1$

$$\int_{0}^{\infty} \frac{\sin s}{s}\, ds = \frac{\pi}{2}$$

take $s = x$

$$\int_{0}^{\infty} \frac{\sin x}{x}\, dx = \frac{\pi}{2}$$

③ Find the fourier transform of $F(x) = \begin{cases} 1, & |x| < 1 \\ 0, & |x| \geq 1 \end{cases}$ and hence find $\int_{0}^{\infty} \frac{\sin x}{x}\, dx$

Given

$$F(x) = \begin{cases} 1, & |x| < 1 \\ 0, & |x| \geq 1 \end{cases}$$

$$F(x) = \begin{cases} 1, & -1 < x < 1 \\ 0, & x \geq 1 \end{cases}$$

$$\text{WKT } F[F(x)] = \int_{-\infty}^{\infty} F(x)\, e^{isx}\, dx$$

$$= \int_{-\infty}^{-1} F(x)\, e^{isx}\, dx + \int_{-1}^{1} F(x)\, e^{isx}\, dx + \int_{1}^{\infty} F(x)\, e^{isx}\, dx$$

$$= 0 + \int_{-1}^{1} F(x) e^{isx} \, dx + 0$$

$$F[F(x)] = \int_{-1}^{1} e^{isx} \, dx$$

$$= \left[\frac{e^{isx}}{is} \right]_{-1}^{1}$$

$$= \frac{1}{is} [e^{is} - e^{-is}]$$

$$= \frac{1}{is} [\cos(s) + i\sin(s) - \cos(s) + i\sin(s)]$$

$$= \frac{1}{\cancel{i}s} [2\cancel{i}\sin(s)]$$

$$= \frac{2}{s} [\sin s]$$

WKT

$$F^{-1}[f(s)] = \frac{1}{2\pi} \int_{-\infty}^{\infty} f(s) e^{-isx} \, ds = F(x)$$

$$= \int_{-\infty}^{\infty} \frac{2}{s} \sin s \, e^{-isx} \, ds = 2\pi F(x) = 2\pi \begin{cases} 1, & -1 < x < 1 \\ 0, & x > 1 \end{cases}$$

Let $x = 0$

(1) $\Rightarrow \int_{-\infty}^{\infty} \frac{2}{s} \sin ds = 2\pi(1)$

$$= \int_{-\infty}^{\infty} \frac{\cancel{2}}{s} \sin s \, ds = \cancel{2}\pi(1)$$

$$= 2 \int_{-\infty}^{\infty} \frac{\sin s}{s} \, ds = \pi$$

$$= \int_{0}^{\infty} \frac{\sin s}{s} \, ds = \frac{\pi}{2}$$

when $s = x$

$$= \int_{0}^{\infty} \frac{\sin x}{x} \, dx = \pi/2 \quad //$$

(4) Find the fourier transform of $e^{-|x|}$

$$F(x) = \begin{cases} e^{-(-x)}, & x < 0 \\ e^{-x}, & x > 0 \end{cases}$$

$$F(x) = \begin{cases} e^{x}, & x < 0 \\ e^{-x} & x > 0 \end{cases}$$

WKT

$$F[F(x)] = \int_{-\infty}^{\infty} F(x) e^{isx} dx.$$

$$= \int_{-\infty}^{0} F(x) e^{isx} dx + \int_{0}^{\infty} F(x) e^{isx} dx$$

$$= \int_{-\infty}^{0} e^{x} e^{isx} dx + \int_{0}^{\infty} e^{-x} e^{isx} dx$$

$$= \int_{-\infty}^{0} e^{x+isx} dx + \int_{0}^{\infty} e^{-x+isx} dx$$

$$= \int_{-\infty}^{0} e^{[1+is]x} dx + \int_{0}^{\infty} e^{-[1-is]x} dx$$

$$= \left[\frac{e^{(1+is)x}}{1+is}\right]_{-\infty}^{0} + \left[\frac{e^{-(1-is)x}}{1-is}\right]_{0}^{\infty}$$

$$= \frac{1}{1+is}\left[e^{(1+is)x}\right]_{-\infty}^{0} - \frac{1}{1-is}\left[e^{-(1-is)x}\right]_{0}^{\infty}$$

$$= \frac{1}{1+is}[1-0] - \frac{1}{1-is}[0-1]$$

$$= \frac{1}{1+is} + \frac{1}{1-is}$$

$$= \frac{1-\cancel{is}+1+\cancel{is}}{1^2+i^2s^2}$$

$$= \frac{2}{1+s^2} \quad //$$

(5) Find the fourier transform $F(x) = \begin{cases} a^2 - x^2, & |x| \le a \\ 0, & |x| > a \end{cases}$

hence show that (i) $\displaystyle\int_{0}^{\infty} \frac{\sin x - x\cos x}{x^3} dx = \frac{\pi}{4}$

(ii) $\int_0^\infty \frac{x\cos x - \sin x}{x^3} \cos\left(\frac{x}{2}\right) dx = \frac{-3\pi}{16}$

Given $F(x) = \begin{cases} a^2 - x^2, & -a \le x \le a \\ 0, & x > a \end{cases}$

WKT $F[F(x)] = \int_{-\infty}^{\infty} e^{isx} F(x)\, dx$

$$= \int_{-\infty}^{a} e^{isx} F(x)\, dx + \int_{-a}^{a} e^{isx} F(x)\, dx + \int_{a}^{\infty} e^{isx} F(x)\, dx$$

$$= \int_{-a}^{a} (a^2 - x^2) e^{isx}\, dx$$

$$= (a^2 - x^2)\int_{-a}^{a} e^{isx}\, dx - \int_{-a}^{a} \left[-2x \int e^{isx} dx\right] dx$$

$$= \frac{1}{is}(0 - 0) + \frac{2}{is}\int_{-a}^{a} x e^{isx}\, dx$$

$$= \frac{2}{is}\left\{ x\int_{-a}^{a} e^{isx}\, dx - \int_{-a}^{a} \left[1 \cdot \int e^{isx} dx\right] dx \right\}$$

$$= \frac{2}{is}\left\{ \frac{1}{is}\left[x e^{isx}\right]_{-a}^{a} - \frac{1}{i^2 s^2}\left[e^{isx}\right]_{-a}^{a} \right\}$$

$$= \frac{2}{i^2 s^2}\left[x e^{isx}\right]_{-a}^{a} - \frac{2}{i^3 s^3}\left[e^{isx}\right]_{-a}^{a}$$

$$= -\frac{2}{s^2}\left[a e^{ias} + a e^{-ias}\right] + \frac{2}{is^3}\left[e^{ias} - e^{-ias}\right]$$

$$= -\frac{2a}{s^2}\left[e^{ias} + e^{-ias}\right] + \frac{2}{is^3}\left[e^{ias} - e^{-ias}\right]$$

$$= -\frac{2a}{s^2}\left[2\cos(as)\right] + \frac{2}{is^3}\left[2i\sin(as)\right]$$

$$= \frac{-4a\cos(as)}{s^2} + \frac{4\sin(as)}{s^3}$$

$$= \frac{4\sin(as) - 4as\cos(as)}{s^3}$$

$$f(s) = \frac{4}{s^3}\left[\sin(as) - as\cos(as)\right]$$

We know that

The fourier inverse transform is

$$F^{-1}[f(s)] = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-isx} f(s)\,ds = F(x)$$

$$\Rightarrow \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-isx}\frac{4}{s^3}[\sin(as) - as\cos(as)ds] = \begin{cases} a^2-x^2, & |x| \le a \\ 0, & |x| > a \end{cases}$$

$$\Rightarrow \int_{-\infty}^{\infty} e^{-isx}\frac{[\sin(as) - as\cos(as)]}{s^3}\,ds = \frac{\pi}{2}\begin{cases} a^2-x^2, & |x| \le a \\ 0, & |x| > a \end{cases}$$

$$\Rightarrow \int_{-\infty}^{\infty}[\cos(sx) - i\sin(sx)]\frac{[\sin(as) - as\cos(as)]}{s^3}\,ds = \frac{\pi}{2}\begin{cases} a^2-x^2, & |x| \le a \\ 0, & |x| > a \end{cases}$$

But $x = 0$

$$(1) \Rightarrow \int_{-\infty}^{\infty}\frac{\sin(as) - as\cos(as)}{s^3}\,ds = \frac{\pi}{2}a^2 \quad - (2)$$

$$\int_{-\infty}^{\infty}\frac{\sin s - s\cos s}{s^3}\,ds = \frac{\pi}{2}, \quad \text{for } a=1$$

$$\Rightarrow 2\int_{0}^{\infty}\frac{\sin s - s\cos s}{s^3}\,ds = \frac{\pi}{2}$$

$$\int_{0}^{\infty}\frac{\sin s - s\cos s}{s^3}\,ds = \frac{\pi}{4}$$

$$\int_{0}^{\infty}\frac{\sin x - x\cos x}{x^3}\,dx = \frac{\pi}{4} \quad \text{for } s = x.$$

(ii) Let $x = a/2$

$$(1) \Rightarrow \int_{-\infty}^{\infty}\frac{\sin(as) - s\cos(as)}{s^3}\cos\left(\frac{as}{2}\right) = \frac{\pi}{2}\left(a^2 - \frac{a^2}{4}\right) = \frac{3\pi a^2}{8}$$

$$\Rightarrow \int_{-\infty}^{\infty} \frac{\sin s - s\cos s}{s^3} \cos\left(\frac{s}{2}\right) ds = \frac{3\pi}{8} \text{ for } a=1$$

$$\Rightarrow 2\int_{0}^{\infty} \frac{\sin s - s\cos s}{s^3} \cos\left(\frac{s}{2}\right) ds = \frac{3\pi}{8}$$

$$\Rightarrow \int_{0}^{\infty} \frac{\sin s - s\cos s}{s^3} \cos\left(\frac{s}{2}\right) ds = \frac{3\pi}{16}$$

$$\Rightarrow \int_{0}^{\infty} \frac{s\cos s - \sin s}{s^3} \cos\left(\frac{s}{2}\right) ds = -\frac{3\pi}{16}$$

$$\Rightarrow \int_{0}^{\infty} \frac{x\sin x - \sin x}{x^3} \cos\left(\frac{x}{2}\right) dx = \frac{-3\pi}{16} \quad //$$

⑥ Find the fourier transform $F(x) = \begin{cases} 1-x^2, & |x| \le 1 \\ 0, & |x| > 1 \end{cases}$ hence show that (i) $\int_0^\infty \frac{\sin x - x\cos x}{x^3} dx = \frac{\pi}{4}$

(ii) $\int_0^\infty \frac{x\cos x - \sin x}{x^3} \cos\left(\frac{x}{2}\right) dx = \frac{-3\pi}{16}$

Given $F(x) \begin{cases} 1-x^2, & -1 \le x \le 1 \\ 0, & x > 1 \end{cases}$

WKT

$$F[F(x)] = \int_{-\infty}^{\infty} e^{isx} F(x) dx$$

$$= \int_{-\infty}^{-1} e^{isx} F(x) dx + \int_{-1}^{1} e^{isx} F(x) dx + \int_{1}^{\infty} e^{isx} F(x) dx$$

$$= \int_{-1}^{1} (1^2 - x^2) e^{isx} dx$$

$$= (1-x^2) \int_{-1}^{1} e^{isx} dx - \int_{-1}^{1} [-2x \int e^{isx} dx] dx$$

$$= \frac{1}{is} \left[(1-x^2) e^{isx} \right]_{-1}^{1} + \frac{2}{is} \int_{-1}^{1} x e^{isx} dx$$

$$= \frac{1}{is} (0-0) + \frac{2}{is} \int_{-1}^{1} x e^{isx} dx$$

$$= \frac{2}{is} \left\{ x \int_{-1}^{1} e^{isx} dx - \int_{-1}^{1} [1 \int e^{isx} dx] dx \right\}$$

$$= \frac{2}{is}\left\{ \frac{1}{is}\left[xe^{isx}\right]_{-1}^{1} - \frac{1}{i^2s^2}\left[e^{isx}\right]_{-1}^{1} \right\}$$

$$= \frac{2}{i^2s^2}\left[xe^{isx}\right]_{-1}^{1} - \frac{2}{i^3s^3}\left[e^{isx}\right]_{-1}^{1}$$

$$= -\frac{2}{s^2}\left[e^{is}+e^{-is}\right] + \frac{2}{is^3}\left[e^{is}-e^{-is}\right]$$

$$= \frac{-2}{s^2}\left[2\cos(s)\right] + \frac{2}{is^3}\left[2i\sin(s)\right]$$

$$= \frac{-4\cos(s)}{s^2} + \frac{4\sin(s)}{s^3}$$

$$= \frac{4\sin(s) - 4s\cos(s)}{s^3}$$

$$f(s) = \frac{4}{s^3}\left[\sin(s) - s\cos(s)\right]$$

WKT

The Fourier inverse transform is

$$F^{-1}[f(s)] = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-isx} f(s)\,ds = F(x)$$

$$\Rightarrow \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-isx}\frac{4}{s^3}\left[\sin s - s\cos s\right]ds = \begin{cases} 1-x^2, & |x| \le 1 \\ 0, & |x| > 1 \end{cases}$$

$$\Rightarrow \int_{-\infty}^{\infty} e^{-isx}\left[\frac{\sin s - s\cos s}{s^3}\right]ds = \frac{\pi}{2}\begin{cases} 1-x^2, & |x| \le 1 \\ 0, & |x| > 1 \end{cases}$$

$$\Rightarrow \int_{-\infty}^{\infty}\left[\cos(sx) - i\sin(sx)\right]\frac{\left[\sin(s) - s(\cos s)\right]}{s^3}ds = \frac{\pi}{2}\begin{cases} 1-x^2, & |x| \le 1 \\ 0, & |x| > 1 \end{cases}$$

$$\Rightarrow \int_{-\infty}^{\infty}\frac{\sin s - s\cos s}{s^3}\cos(sx)\,ds = \frac{\pi}{2}\begin{cases} 1-x^2, & |x| \le 1 \\ 0, & |x| > 1 \end{cases} \quad -①$$

But $x = 0$

$$\therefore ① \Rightarrow \int_{-\infty}^{\infty}\frac{\sin s - s\cos s}{s^3}ds = \frac{\pi}{2} \quad -②$$

$$\Rightarrow \int_{-\infty}^{\infty}\frac{\sin s - s\cos s}{s^3}ds = \frac{\pi}{2}$$

$$\Rightarrow 2\int_{0}^{\infty}\frac{\sin s - s\cos s}{s^3}ds = \frac{\pi}{2}$$

$$\Rightarrow \int_0^\infty \frac{\sin s - s\cos s}{s^3}\, ds = \frac{\pi}{4}$$

$$\Rightarrow \int_0^\infty \frac{\sin x - x\cos x}{x^3}\, ds = \frac{\pi}{4} \quad \text{for } s = x$$

(ii) Let $x = \frac{1}{2}$

$$① \Rightarrow \int_{-\infty}^{\infty} \frac{\sin s - s\cos s}{s^3} \cos\left(\frac{s}{2}\right) ds = \frac{\pi}{2}\left(1^2 - \frac{1^2}{4}\right) = \frac{3\pi}{8}$$

$$\Rightarrow \int_{-\infty}^{\infty} \frac{\sin s - s\cos s}{s^3} \cos\left(\frac{s}{2}\right) ds = \frac{3\pi}{8}$$

$$\Rightarrow 2\int_0^{\infty} \frac{\sin s - s\cos s}{s^3} \cos\left(\frac{s}{2}\right) ds = \frac{3\pi}{8}$$

$$\Rightarrow \int_0^{\infty} \frac{\sin s - s\cos s}{s^3} \cos\left(\frac{s}{2}\right) ds = \frac{3\pi}{16}$$

$$\Rightarrow \int_0^{\infty} \frac{s\cos s - \sin s}{s^3} \cos\left(\frac{s}{2}\right) ds = -\frac{3\pi}{16}$$

$$\Rightarrow \int_0^{\infty} \frac{x\cos x - \sin x}{x^3} \cos\left(\frac{x}{2}\right) dx = \frac{-3\pi}{16} \quad \text{for } s = x$$

⑦ Find the fourier transform of $F(x) = \begin{cases} 1 - |x|, & |x| \le 1 \\ 0 & |x| > 1 \end{cases}$

$$F(x) = \begin{cases} 1 - |x|, & -1 \le x \le 1 \\ 0, & x > 1 \end{cases}$$

$$F(x) = \begin{cases} 1 + x, & -1 \le x < 0 \\ 1 - x, & 0 \le x < 1 \\ 0, & x > 1 \end{cases}$$

WKT

$$F[F(x)] = \int_{-\infty}^{\infty} F(x) e^{isx} dx$$

$$f(s) = \int_{-\infty}^{-1} F(x) e^{isx} dx + \int_{-1}^{0} F(x) e^{isx} dx + \int_0^1 F(x) e^{isx} dx + \int_1^\infty F(x) e^{isx} dx$$

$$= \int_{-1}^{0} (1+x) e^{isx} dx + \int_0^1 (1-x) e^{isx} dx \quad -①$$

$$\int_{-1}^{0} (1+x) e^{isx} dx = \left\{ (1+x) \int_{-1}^{0} e^{isx} dx - \int_{-1}^{0} \left[1 \int e^{isx} dx\right] dx \right\}$$

$$= \frac{1}{is}\left[(1+x)e^{isx}\right]_{-1}^{0} - \frac{1}{is^2}\left(e^{isx}\right)_{-1}^{0}$$

$$= \frac{1}{is}\left[(1+x)e^{isx}\right]_{-1}^{0} + \frac{1}{s^2}\left(e^{isx}\right)_{-1}^{0}$$

$$= \frac{1}{is}[1-0] + \frac{1}{s^2}[1-e^{-is}]$$

$$= \frac{1}{is} + \frac{1}{s^2}[1-e^{-is}]$$

$$= \int_0^1 (1-x)e^{isx}\,dx = \frac{1}{is}\left[(1-x)e^{isx}\right]_0^1 - \frac{1}{s^2}\left[e^{isx}\right]_0^1$$

$$= \frac{1}{is}[0-1] - \frac{1}{s^2}[e^{is}-1]$$

$$= -\frac{1}{is} - \frac{1}{s^2}[e^{is}-1]$$

$$f(s) = \cancel{\frac{1}{is}} + \frac{1}{s^2}[1-e^{-is}] - \cancel{\frac{1}{is}} - \frac{1}{s^2}[e^{is}-1]$$

$$f(s) = \frac{1}{s^2}[1-e^{-is}-e^{is}+1]$$

$$= \frac{1}{s^2}[2-(e^{is}+e^{-is})]$$

$$= \frac{1}{s^2}[2-2\cos s]$$

$$= \frac{2}{s^2}[1-\cos s]$$

WKT

$$F^{-1}[F(s)] = \frac{1}{2\pi}\int_{-\infty}^{\infty} f(s)e^{-isx}\,ds = F(x)$$

$$= \int_{-\infty}^{\infty} \frac{2}{s^2}[1-\cos s]e^{-isx}\,dx = 2\pi\begin{cases}1-|x|, & -1\le x\le 1\\ 0, & x>1\end{cases} \quad -(1)$$

Let $x=0$

$$(1) \Rightarrow \cancel{2}\int_{-\infty}^{\infty} \frac{1-\cos s}{s^2}\,ds = 2\pi(1) = \cancel{2}\pi$$

$$= \int_{-\infty}^{\infty} \frac{1-\cos s}{s^2}\,ds = \pi$$

$$= \int_{-\infty}^{\infty} \frac{2\sin^2(s/2)}{s^2}\, ds = \pi$$

$$= 2\int_{-\infty}^{\infty} \frac{\sin^2(s/2)}{4\times(s/2)^2}\, ds = \pi$$

$$= \int_{-\infty}^{\infty} \frac{\sin^2(s/2)}{(s/2)^2}\, ds = 2\pi \quad -(2)$$

Let $\frac{s}{2} = u \Rightarrow s = 2u \Rightarrow ds = 2du$

$$(2) \Rightarrow \int_{-\infty}^{\infty} \frac{\sin^2 u}{u^2}\, \cancel{2} du = \cancel{2}\pi$$

$$\Rightarrow \int_{-\infty}^{\infty} \frac{\sin^2 u}{u^2}\, du = \pi$$

$$\Rightarrow 2\int_{0}^{\infty} \frac{\sin^2 u}{u^2}\, du = \pi$$

$$\Rightarrow \int_{0}^{\infty} \frac{\sin^2 u}{u^2}\, du = \frac{\pi}{2}$$

$$\Rightarrow \int_{0}^{\infty} \frac{\sin^2 t}{t^2}\, dt = \frac{\pi}{2} \quad \text{for } u = t$$

(8) Find the inverse fourier transform of e^{-s^2}.

Let $f(s) = e^{-s^2}$

$$F(x) = F^{-1}[f(s)] = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-isx} f(s)\, ds$$

$$F(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-isx} e^{-s^2}\, ds$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-(s^2+isx)}\, ds$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-\left[s^2 + 2(s)\left(\frac{ix}{2}\right) + \left(\frac{ix}{2}\right)^2 - \left(\frac{ix}{2}\right)^2\right]}\, ds$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-\left[(s+\frac{ix}{2})^2 - \frac{i^2x^2}{4}\right]} ds$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-\left[(s+\frac{ix}{2})^2 + \frac{x^2}{4}\right]} ds$$

$$F(x) = \frac{e^{-x^2/4}}{2\pi} \int_{-\infty}^{\infty} e^{-(s+\frac{ix}{2})^2} ds$$

Let $s+\frac{ix}{2} = u$

$\Rightarrow ds = du$

$\therefore$ ① $\Rightarrow F(x) = \dfrac{e^{-x^2/4}}{2\pi} \displaystyle\int_{-\infty}^{\infty} e^{-u^2} du$

$$F(x) = \frac{e^{-x^2/4}}{2\pi} \sqrt{\pi}$$

$$F(x) = \frac{1}{2\sqrt{\pi}\cancel{\sqrt{\pi}}} e^{-x^2/4} \cancel{\sqrt{\pi}}$$

$$\Rightarrow F(x) = \frac{1}{2\sqrt{\pi}} e^{-x^2/4}$$

⑨ Find the fourier cosine & sin transform of $F(x) = \begin{cases} x, & 0<x<1 \\ 2-x, & 1<x<2 \\ 0, & x>2 \end{cases}$

Given $F(x) = \begin{cases} x, & 0<x<1 \\ 2-x, & 1<x<2 \\ 0, & x>2 \end{cases}$

WKT the fourier sin transform of $F(x)$ is

$$F_S[F(s)] = \int_0^{\infty} F(x)\sin(sx)\,dx$$

$$f_s(s) = \int_0^1 F(x)\sin(sx)\,dx + \int_1^2 F(x)\sin(sx)\,dx + \int_2^{\infty} F(x)\sin(sx)\,dx$$

$$= \int_0^1 x\sin(sx)\,dx + \int_1^2 (2-x)\sin(sx)\,dx$$

$$f_s(s) = I_1 + I_2$$

$$I_1 = \int_0^1 x\sin(sx)\,dx$$

$$= x\int_0^1 \sin(sx)\,dx - \int_0^1 [1 \int \sin(sx)dx]\,dx$$

$$= -\left[\frac{x\cos sx}{s}\right]_0^1 + \frac{1}{s^2}\sin sx\Big]_0^1$$

$$= -\left[\frac{\cos s - 0}{s}\right] + \frac{1}{s^2}[\sin s - 0]$$

$$I_1 = \frac{-\cos s}{s} + \frac{1}{s^2}\sin s$$

$$I_2 = \int_1^2 (2-x)\sin(sx)\,dx$$

$$= \left\{(2-x)\int_1^2 \sin(sx)\,dx - \int_1^2 [-1 \int \sin(sx)dx]\,dx\right\}$$

$$= \left[\frac{(2-x)\cos sx}{s}\right]_1^2 - \frac{1}{s}\int_1^2 \cos sx\,dx$$

$$= \left[\frac{(2-x)\cos sx}{s}\right]_1 \int_1^2 \frac{1}{s^2}[\sin sx]_1^2$$

$$= -\left[\frac{0 - \cos s}{s}\right] - \frac{1}{s^2}[\sin 2s - \sin s]$$

$$I_2 = \frac{\cos s}{s} - \frac{1}{s^2}[\sin 2s + \frac{1}{s^2}[\sin s]$$

$$f_s(s) = -\frac{\cancel{\cos s}}{\cancel{s}} + \frac{1}{s^2}\sin s + \frac{\cancel{\cos s}}{\cancel{s}} - \frac{1}{s^2}[\sin 2s + \frac{1}{s^2}\sin s]$$

$$f_s(s) = \frac{2}{s^2}\sin s - \frac{1}{s^2}\sin 2s \text{ //}$$

WKT the fourier cosin transform of $F(x)$ is

$$F_c[F(x)] = \int_0^\infty F(x)\cos(sx)\,dx$$

$$f_c(s) = \int_0^1 F(x)\cos(sx)dx + \int_1^2 F(x)\cos(sx)\,dx + \int_2^\infty F(x)\cos(sx)dx$$

$$= \int_0^1 x\cos(sx)\,dx + \int_1^2 (2-x)\cos(sx)\,dx$$

$$f_c(s) = I_1 + I_2$$

$$I_1 = \int_0^1 x\cos(sx)\,dx$$

$= x\int_0^1 \cos(sx)dx - \int_0^1 [1 \int \cos(sx)dx]dx$

$= \left[\frac{x\sin sx}{s}\right]_0^1 + \frac{1}{s^2}\cos sx\Big]_0^1$

$= \frac{\sin s}{s} - 0 + \frac{1}{s^2}[\cos sx - 1]$

$= \frac{\sin s}{s} + \frac{1}{s^2}\cos sx - \frac{1}{s^2}$

$I_2 = \int_1^2 (2-x)\cos(sx)\,dx$

$= (2-x)\int_1^2 \cos(sx)dx - \int_1^2 [(-1)\int \cos(sx)dx]dx$

$= -\left[\frac{(2-x)\sin sx}{s}\right]_1^2 - \int_1^2 \frac{1}{s^2}\sin sx]_1^2 dx$

$= \frac{x-2}{s}[\sin sx]_1^2 - \frac{1}{s^2}[\cos sx]_1^2$

$= \frac{\sin s}{s} - \frac{\cos 2s}{s^2} + \frac{\cos s}{s^2}$

$F_c[F(x)] = \frac{1}{s}\sin s + \frac{1}{s^2}\cos s - \frac{1}{s^2} + \frac{\sin s}{s} - \frac{\cos 2s}{s^2} + \frac{\cos s}{s^2}$

$= \frac{2}{s}\sin s + \frac{2}{s^2}\cos s - \frac{1}{s^2} + \frac{\cos 2s}{s^2}$ //

(10) Find the Fourier sin and cosin of $F(x) = \begin{cases} 4x, & 0 < x < 1 \\ 4-x, & 1 \le x < 4 \\ 0, & x \ge 4 \end{cases}$

Given $F(x) = \begin{cases} 4x, & 0 < x < 1 \\ 4-x, & 1 \le x < 4 \\ 0, & x \ge 4 \end{cases}$

WKT the fourier cosin transform of $F(x)$ is

$F_c[F(x)] = \int_0^\infty F(x)\cos(sx)\,dx$

$= \int_0^1 F(x)\cos(sx)dx + \int_1^4 F(x)\cos(sx)dx + \int_4^\infty F(x)\cos(sx)dx$

$= 4\int_0^1 x\cos(sx)dx + \int_1^4 (4-x)\cos(sx)dx$ — (1)

$= \frac{4x\sin(sx)}{s} + \frac{4}{s^2}[\cos(sx)]_0^1$

$= \frac{4}{s}[4x \sin(sx)]_0^1 + \frac{4}{s}[\cos(sx)]_0^1$

$= \frac{4}{s}[\sin(s) - 0] + \frac{4}{s^2}[\cos(s) - 1]$

$= \frac{4}{s}\sin s + \frac{4}{s^2}\cos s - \frac{4}{s^2}$

$\int_1^4 (4-x)\cos(sx)dx = (4-x)\int_1^4 \cos(sx)dx - \int_1^4 -1\left[\int\cos(sx)dx\right]dx$

$= \frac{1}{s}[(4-x)\sin(sx)]_1^4 - \frac{1}{s^2}(\cos(sx))_1^4$

$= \frac{1}{s^2}\cos s - \frac{1}{s^2}\cos 4s - \frac{1}{s}\sin s$

$f_c(s) = \frac{1}{s}\sin s + \frac{4}{s^2}\cos s - \frac{4}{s^2} + \frac{1}{s^2}\cos - \frac{1}{s^2}\cos 4s - \frac{3}{s}\sin s$

$f_c(s) = \frac{1}{s}\sin s + \frac{5}{s^2}\cos s - \frac{1}{s^2}\cos 4s - \frac{4}{s^2}$ //

(ii) $F_s(F(x)] = \int_0^\infty F(x)\sin(sx)dx$

$= \int_0^1 F(x)\sin(sx)dx + \int_1^4 F(x)\sin(sx)dx + \int_4^\infty F(x)\sin(sx)dx$

$= \int_0^1 4x\sin(sx)dx + \int_1^4 (4-x)\sin(sx)dx$

$= 4x\int_0^1 \sin(sx)dx - \int_0^1 4\left[\int\sin(sx)dx\right]dx$

$= -\frac{1}{s}[4x\cos(sx)]_0^1 + \frac{4}{s^2}[\sin(sx)]_0^1$

$= -\frac{4}{s}[\cos(s) - 1] + \frac{4}{s^2}[\sin(s) - 0]$

$= -\frac{4}{s}\cos(s) + \frac{4}{s^2}\sin(s)$

$= (4-x)\int_1^4 \sin(sx)dx - \int_1^4 -1\left[\int\sin(sx)dx\right]dx$

$$= \frac{-1}{s}\left[(4-x)\cos(sx)\right]_1^4 + \frac{1}{s^2}\left[\sin sx\, dx\right]_1^4$$

$$= \frac{-1}{s}\left[0 - 3\cos(x)\right] - \frac{1}{s^2}\left[\sin(4s) - \sin(s)\right]$$

$$= \frac{3\cos s}{s} - \frac{1}{s^2}\sin(4s) + \frac{1}{s^2}\sin(s)$$

$$= -\frac{4}{s}\cos(s) + \frac{4}{s^2}\sin(s) + \frac{3\cos s}{s} - \frac{1}{s^2}\sin(4s) + \frac{1}{s^2}\sin(s)$$

$$F_s(s) = \frac{5}{s^2}\sin(s) - \frac{1}{s}\cos(s) - \frac{1}{s^2}\sin 4s \;//$$

(11) Find the fourier sin transform of $e^{-|x|}$, and hence Show that $\int_0^{\infty} \frac{x \sin mx}{1+x^2}\, dx = \frac{\pi}{2} e^{-m}$ for $m > 0$.

$$F(x) = e^{-|x|}$$

WKT $|x| = \begin{cases} x, & x > 0 \\ -x, & x < 0 \end{cases}$

$$F(x) = \begin{cases} e^{-x}, & \text{for } x > 0 \\ e^{x}, & \text{for } x < 0 \end{cases}$$

Fourier sin transform of $F(x)$ is

$$F_s[F(x)] = f_s(s) = \int_0^{\infty} F(x)\sin(sx)\, dx$$

$$= \int_0^{\infty} e^{-x}\sin(sx)\, dx$$

$$F_s[F(x)] = \left\{\frac{e^{-x}}{(-1)^2+s^2}\left[-1\sin(sx) - s\cos(sx)\right]\right\}_0^{\infty}$$

$$= \frac{-1}{1+s^2}\left[e^{-x}\sin(sx) + s\cos(sx)\right]_0^{\infty}$$

$$= \frac{-1}{1+s^2}\left\{0 - 1\left[0 + s(1)\right]\right\}$$

$$= \frac{-1}{1+s^2}(-s)$$

$$\Rightarrow f_s(s) = \frac{s}{s^2+1}$$

WKT inverse fourier sin transform.

$$F^{-1}[f_s(s)] = \frac{2}{\pi}\int_0^\infty f_s(s)\sin(sx)\,ds = F(x)$$

$$= \int_0^\infty \frac{s}{s^2+1}\sin(sx)\,ds = \frac{\pi}{2}e^{-x} \quad (1)$$

Let $x = m > 0$

$$(1) \Rightarrow \int_0^\infty \frac{s\sin(ms)}{1+s^2}\,ds = \frac{\pi}{2}e^{-m}$$

Let $s = x$

$$(2) \Rightarrow \int_0^\infty \frac{x\sin(mx)}{1+x^2}\,dx = \frac{\pi}{2}e^{-m}$$

(12) Find the Fourier cosin transform of $e^{-2x}+4e^{-3x}$

Let $F(x) = e^{-2x}+4e^{-3x}$

$$F_c[F(x)] = \int_0^\infty F(x)\cos(sx)\,dx$$

$$= \int_0^\infty (e^{-2x}+4e^{-3x})\cos(sx)\,dx$$

$$= \int_0^\infty e^{-2x}\cos(sx)\,dx + 4\int_0^\infty e^{-3x}\cos(sx)\,dx.$$

$$= \int_0^\infty e^{-2x}\cos(sx)\,dx = \frac{e^{-2x}}{(-2)^2+s^2}\Big[-2\cos(sx)+s\sin(sx)\Big]_0^\infty$$

$$= \frac{e^{-2x}}{s^2+4}\Big[-2\cos(sx)+s\sin(sx)\Big]_0^\infty$$

$$= \{0\} - \left\{\frac{1}{s^2+4}(-2)\right\}$$

$$= \frac{2}{s^2+4}$$

$$\int_0^\infty e^{-3x}\cos(sx)\,dx = \frac{e^{-3x}}{s^2+9}\left[-3\cos(sx)+s\sin(sx)\right]_0^\infty$$

$$= \frac{e^{-3x}}{s^2+9}\left[-3\cos(3x)+s\sin(sx)\right]_0^\infty$$

$$= [0] - \left\{\frac{-3}{s^2+9}\right\}$$

$$= \frac{3}{s^2+9}$$

$$f(s) = \frac{2}{s^2+4} + 4\,\frac{3}{s^2+9}$$

$$f_c(s) = \frac{2}{s^2+4} + \frac{12}{s^2+9}$$

(13) Find the Fourier cosin and sin transform of $2e^{-3x}+3e^{-2x}$

$$F(x) = 2e^{-3x} + 3e^{-2x}$$

① $$F_c[F(x)] = \int_0^\infty F(x)\cos(sx)\,dx$$

$$= \int_0^\infty (2e^{-3x} + 3e^{-2x})\cos(sx)\,dx$$

$$F_c[f(x)] = \int_0^\infty 2e^{-3x}\cos(sx)\,dx + \int_0^\infty 3e^{-2x}\cos(sx)\,dx \quad -①$$

$$\Rightarrow 2\int_0^\infty e^{-3x}\cos(sx)\,dx = \frac{e^{-3x}}{(-3)^2+s^2}\left[-3\cos(sx)+s\sin(sx)\right]_0^\infty$$

$$= \frac{e^{-3x}}{s^2+9}\left[-3\cos(sx)+s\sin(sx)\right]_0^\infty$$

$$= 0 - \left\{\frac{1}{s^2+9}(-3)\right\}$$

$$= 2\,\frac{3}{s^2+9}$$

$$= \frac{6}{s^2+9}$$

$$3\int_0^\infty e^{-2x}\cos(sx)\,dx = \frac{e^{-2x}}{s^2+4}\left[-2\cos(sx)+s\sin(sx)\right]_0^\infty$$

$$= \frac{e^{-2x}}{s^2+4}\left[-2\cos(sx) + s\sin(sx)\right]_0^\infty$$

$$= 0 - \left\{\frac{-2}{s^2+4}\right\}$$

$$= 3 \cdot \frac{2}{s^2+4}$$

$$= \frac{6}{s^2+4}$$

$$f_c(s) = \frac{6}{s^2+9} + \frac{6}{s^2+4} \,//$$

② WKT

$$f_s[F(x)] = \int_0^\infty F(x)\sin(sx)\,dx$$

$$= \int_0^\infty (2e^{-3x} + 3e^{-2x})\sin(sx)\,dx$$

$$= 2\int_0^\infty e^{-3x}\sin(sx)\,dx + 3\int e^{-2x}\sin(sx)\,dx \quad -①$$

$$f_s(s) = 2\int_0^\infty e^{-3x}\sin(sx)\,dx = 2 \cdot \frac{e^{-3x}}{s^2+9}\left[-3\sin(sx) - s\cos(sx)\right]_0^\infty$$

$$= 0 - \frac{1}{s^2+9}\{0 - s\}$$

$$= \frac{2s}{s^2+9}$$

$$= 3\int_0^\infty e^{-2x}\sin(sx)\,dx = \frac{e^{-2x}}{s^2+4}\left[-2\sin(sx) - s\cos(sx)\right]_0^\infty$$

$$= 0 - \frac{1}{s^2+4}[0 - s]$$

$$= \frac{s}{s^2+4}$$

$$= \frac{3s}{s^2+4}$$

$$f_s(s) = \frac{2s}{s^2+9} + \frac{3s}{s^2+4} \,//$$

(14) Find the Fourier cosin and sin transform of e^{-ax}

Let $F(x) = e^{-ax}$

WKT

① $F_c[F(x)] = \int_0^\infty F(x)\cos(sx)\,dx$

$= \int_0^\infty e^{-ax}\cos(sx)\,dx$

$= \frac{e^{-ax}}{(-a)^2+s^2}\left\{[-a\cos(sx)+s\sin(sx)]\right\}_0^\infty$

$= \frac{e^{-ax}}{s^2+a^2}[-a\cos(sx)+s\sin(sx)]_0^\infty$

$= 0 - \frac{1}{s^2+a^2}(-a)$

$f_c(s) = \frac{a}{s^2+a^2}$

② WKT

$F_s[F(x)] = \int_0^\infty F(x)\sin(sx)\,dx$

$= \int_0^\infty e^{-ax}\sin(sx)\,dx$

$= \frac{e^{-ax}}{(-a)^2+s^2}[-a\sin(sx) - s\cos(sx)]_0^\infty$

$= \frac{e^{-ax}}{s^2+a^2}\{-a\sin(sx - s\cos(sx)\}_0$

$= 0 - \frac{1}{s^2+a^2}\{0-s\}$

$f_s(s) = \frac{s}{s^2+a^2}$

(15) Find the Fourier transform of sin & cosin of

$F(x) = \begin{cases} x, & 0 < x < 2 \\ 0, & x \geq 2 \end{cases}$

Given $F(x) = \begin{cases} x, & 0 < x < 2 \\ 0, & x \geq 2 \end{cases}$

WKT

① $F_s[F(x)] = \int_0^\infty F(x)\sin(sx)\,dx$

$= \int_0^2 F(x)\sin(sx)\,dx + \int_2^\infty F(x)\sin(sx)\,dx$

$= \int_0^2 x\sin(sx)\,dx$

$f_s(s) = x\int_0^2 \sin(sx)\,dx - \int_0^2 [1\,[\sin sx\,dx]\,dx$

$= -\left[\frac{x\cos(sx)}{s}\right]_0^2 + \frac{1}{s}\int_0^2 \sin(sx)\,dx$

$= -\left[\frac{2\cos 2s - 0}{s}\right] + \frac{1}{s^2}[\sin(sx)]_0^2$

$= -\frac{2\cos 2s}{s} + \frac{1}{s^2}[\sin 2s - 0]$

$= \frac{1}{s^2}\sin 2s - \frac{2}{s}\cos 2s$

② $F_c[F(x)] = \int_0^\infty F(x)\cos(sx)\,dx$

$F(x) = \int_0^2 F(x)\cos(sx)\,dx + \int_2^\infty F(x)\cos(sx)\,dx$

$= \int_0^2 x\cos(sx)\,dx$

$= x\int_0^2 \cos(sx)\,dx - \int_0^2 [1\int\cos(sx)\,dx]\,dx$

$= \left[\frac{x\sin sx}{x}\right]_0^2 + \frac{1}{s^2}[\cos(sx)]_0^2$

$= \frac{2\sin 2s}{s} - 0 + \frac{1}{s^2}[\cos 2s - 1]$

$f_c(s) = \frac{1}{s^2}\cos 2s + \frac{2}{s}\sin 2s - \frac{1}{s^2}$ //

Z-transforms and Difference Equations:-

$(1-x)^{-1} = 1 + x + x^2 + x^3 + \dots$

$(1+x)^{-1} = 1 - x + x^2 - x^3 + x^4 + \dots$

$(1-x)^{-2} = 1 + 2x + 3x^2 + 4x^4 + \dots$

$(1+x)^{-2} = 1 - 2x + 3x^2 - 4x^3 + \dots$

Definition:-

Suppose $f(n)$ be a function in the variable n, such that the z-transform of $f(n)$ can be defined as

$Z[f(n)] = \sum_{n=0}^{\infty} f(n) z^{-n} = F(z)$, where Z is called the z-transform operator.

Some important results:-

1) $f(n) = 1$

$$Z[1] = \sum_{n=0}^{\infty} 1 \cdot z^{-n}$$

$$= \sum_{n=0}^{\infty} \frac{1}{z^n}$$

$$= \frac{1}{z^0} + \frac{1}{z^1} + \frac{1}{z^2} + \frac{1}{z^3} + \dots$$

$$= 1 + \left(\frac{1}{z}\right) + \left(\frac{1}{z}\right)^2 + \left(\frac{1}{z}\right)^3 + \dots$$

$$= \left(1 - \frac{1}{z}\right)^{-1}$$

$$= \left(\frac{z-1}{z}\right)^{-1}$$

$$\boxed{Z[1] = \frac{z}{z-1}}$$

2) $f(n) = a^n$

$$\therefore Z[a^n] = \sum_{n=0}^{\infty} a^n \cdot z^{-n}$$

$$= \sum_{n=0}^{\infty} \left(\frac{a}{z}\right)^n$$

$$= \left(\frac{a}{z}\right)^0 + \left(\frac{a}{z}\right)^1 + \left(\frac{a}{z}\right)^2 + \cdots$$

$$= 1 + \left(\frac{a}{z}\right) + \left(\frac{a}{z}\right)^2 + \cdots$$

$$= \left(1 - \frac{a}{z}\right)^{-1}$$

$$= \left(\frac{z-a}{z}\right)^{-1}$$

$$\boxed{z[a^n] = \frac{z}{z-a}}$$

for $a = -1$

$$z[(-1)^n] = \frac{z}{z+1}$$

③ $f(n) = n$

$$z[n] = \sum_{n=0}^{\infty} n \cdot z^{-n}$$

$$= 0 \cdot z^0 + 1 \cdot z^{-1} + 2 \cdot z^{-2} + 3 \cdot z^{-3} + 4 \cdot z^{-4} + \cdots$$

$$= 1\left(\frac{1}{z}\right) + 2\left(\frac{1}{z}\right)^2 + 3\left(\frac{1}{z}\right)^3 + 4\left(\frac{1}{z}\right)^4 + \cdots$$

$$= \frac{1}{z}\left[1 + 2\left(\frac{1}{z}\right) + 3\left(\frac{1}{z}\right)^2 + 4\left(\frac{1}{z}\right)^3 + \cdots\right]$$

$$= \frac{1}{z}\left[1 - \frac{1}{z}\right]^{-2}$$

$$= \frac{1}{z}\left[\frac{z-1}{z}\right]^2$$

$$= \frac{1}{z} \frac{z^2}{(z-1)^2}$$

$$\boxed{z[n] = \frac{z}{(z-1)^2}}$$

④ $f(n) = n^2$

WKT $z[n^p] = -z \frac{d}{dz} z[n^{p-1}]$ — ①

$\forall\ p = 2, 3, 4 \cdots$

Let $p = 2$

$$① \Rightarrow z[n^2] = -z \frac{d}{dz} z[n]$$

$$= -z \frac{d}{dz}\left[\frac{z}{(z-1)^2}\right]$$

$$= -z\left[\frac{(z-1)^2(1) - z\,2(z-1)(1)}{(z-1)^4}\right]$$

$$= -z\left[\frac{-z-1}{(z-1)^3}\right]$$

$$= \frac{z(z+1)}{(z-1)^3}$$

$$\boxed{z[n^2] = \frac{z^2+z}{(z-1)^3}}$$

① Find the z transform of $\cos n\theta$ and $\sin n\theta$.

Let $f(n) = e^{in\theta}$

$f(n) = \cos n\theta + i\sin n\theta$

$z[f(n)] = z[\cos n\theta] + i z[\sin n\theta]$ — ①

$z[f(n)] = z[e^{in\theta}]$

$z[f(n)] = z[[e^{i\theta}]^n]$

$$= \frac{z}{z-e^{i\theta}}$$

$$= \frac{z}{z-e^{i\theta}} \times \frac{z-e^{-i\theta}}{z-e^{-i\theta}}$$

$$= \frac{z[z-e^{-i\theta}]}{z^2 - ze^{-i\theta} - ze^{i\theta} + 1}$$

$$= \frac{z^2 - z[\cos\theta - i\sin\theta]}{z^2 - z(2\cos\theta) + 1}$$

$$= \frac{z^2 - z(\cos\theta) + iz\sin\theta}{z^2 - 2z\cos\theta + 1}$$

$$= \frac{z^2 - z\cos\theta}{z^2 - 2z\cos\theta + 1} + i\,\frac{z\sin\theta}{z^2 - 2z\cos\theta + 1}$$

$\therefore z[\cos n\theta] = \dfrac{z^2 - z\cos\theta}{z^2 - 2z\cos\theta + 1}$ $\therefore z[\sin n\theta] = \dfrac{z\sin\theta}{z^2 - 2z\cos\theta + 1}$

(2) Find the z transform of cosh nθ and sinh nθ

WKT $\cosh\theta = \dfrac{e^{\theta}+e^{-\theta}}{2}$

$$\cosh(n\theta) = \frac{e^{n\theta}+e^{-n\theta}}{2}$$

$$z[\cosh(n\theta)] = \frac{1}{2}\, z[e^{n\theta}+e^{-n\theta}]$$

$$= \frac{1}{2}\left\{ z[e^{n\theta}] + z[e^{-n\theta}] \right\}$$

$$= \frac{1}{2}\left\{ z[e^{\theta}]^{n} + z[e^{-\theta}]^{n} \right\}$$

$$= \frac{1}{2}\left\{ \frac{z}{z-e^{\theta}} + \frac{1}{z-e^{-\theta}} \right\}$$

$$= \frac{z}{2}\left[\frac{1}{z-e^{\theta}} + \frac{1}{z-e^{-\theta}} \right]$$

$$= \frac{z}{2}\left[\frac{z-e^{-\theta}+z-e^{\theta}}{(z-e^{\theta})(z-e^{-\theta})} \right]$$

$$= \frac{z}{2}\left[\frac{2z-(e^{\theta}+e^{-\theta})}{z^2-z(e^{\theta}+e^{-\theta})} \right]$$

$$= \frac{z}{2}\left[\frac{2z-2\cosh\theta}{z^2-2z\cosh\theta+1} \right]$$

$$= \frac{z\,[z-\cosh\theta]}{z^2-2z\cosh\theta+1}$$

$$z[\cosh n\theta] = \frac{z^2-z\cosh\theta}{z^2-2z\cosh\theta+1}$$

WKT $\sinh\theta = \dfrac{e^{\theta}-e^{-\theta}}{2} \Rightarrow \sinh(n\theta) = \dfrac{e^{n\theta}-e^{-n\theta}}{2}$

$$z[\sinh(n\theta)] = \frac{1}{2}\, z[e^{n\theta}-e^{-n\theta}]$$

$$= \frac{1}{2}\left\{ z[e^{n\theta}] - z[e^{-n\theta}] \right\}$$

$$= \frac{1}{2}\left\{ z[e^{\theta}]^{n} - z[e^{-\theta}]^{n} \right\}$$

$$= \frac{1}{2}\left[\frac{z}{z-e^{\theta}} - \frac{z}{z-e^{-\theta}} \right]$$

$$= \frac{z}{2}\left[\frac{1}{z-e^{\theta}} - \frac{1}{z-e^{-\theta}}\right]$$

$$= \frac{z}{2}\left[\frac{z-e^{-\theta}-z+e^{\theta}}{(z-e^{\theta})(z-e^{-\theta})}\right]$$

$$= \frac{z}{2}\left[\frac{e^{\theta}-e^{-\theta}}{z^2-2z\cosh\theta+1}\right]$$

$$= \frac{z}{2}\left[\frac{2\sinh\theta}{z^2-2z\cosh\theta+1}\right]$$

$$z[\sinh n\theta] = \frac{z\sinh\theta}{z^2-2z\cosh\theta+1} \quad //$$

$$z\,[f(n)] = F(z)$$

(i) $z[a^n f(n)] = F\left(\frac{z}{a}\right)$

(ii) $z[a^{-n} f(n)] = F(az)$

(iii) $z[a^n n] = \dfrac{az}{(z-a)^2}$ called the daming room.

Find the z transform of the following functions:-

① $\cos\left(\frac{n\pi}{2}+\frac{\pi}{4}\right)$

Let $f(n) = \cos\left(\frac{n\pi}{2}+\frac{\pi}{4}\right)$

$$f(n) = \cos\left(\frac{n\pi}{2}\right)\cos\left(\frac{\pi}{4}\right) - \sin\left(\frac{n\pi}{2}\right)\sin\left(\frac{\pi}{4}\right)$$

$$= \frac{1}{\sqrt{2}}\cos\left(\frac{n\pi}{2}\right) - \frac{1}{\sqrt{2}}\sin\left(\frac{n\pi}{2}\right)$$

$$z[f(n)] = \frac{1}{\sqrt{2}}\, z\left[\cos\left(\frac{n\pi}{2}\right)\right] - \frac{1}{\sqrt{2}}\, z\left[\sin\left(\frac{n\pi}{2}\right)\right]$$

WKT $z[\cos n\theta] = \dfrac{z^2 - z\cos\theta}{z^2-2z\cos\theta+1}$

$$z\left[\cos\left(\frac{n\pi}{2}\right)\right] = \frac{z^2 - z\cos(\pi/2)}{z^2 - 2z\cos\left(\frac{\pi}{2}\right)+1}$$

$$z\left[\cos\left(\frac{n\pi}{2}\right)\right] = \frac{z^2-0}{z^2-2z(0)+1}$$

$$= \frac{z^2}{z^2+1}$$

$$z[\sin(n\theta)] = \frac{z\sin\theta}{z^2-2z\cos\theta+1}$$

$$z\left[\sin\left(\frac{n\pi}{2}\right)\right] = \frac{z\sin(\pi/2)}{z^2-2z\cos\left(\frac{\pi}{2}\right)+1}$$

$$= \frac{z}{z^2-0+1}$$

$$z\left[\sin\left(\frac{n\pi}{2}\right)\right] = \frac{z}{z^2+1}$$

$$① \Rightarrow F(z) = \frac{1}{\sqrt{2}}\,\frac{z^2}{z^2+1} - \frac{1}{\sqrt{2}}\,\frac{z}{z^2+1}$$

$$F(z) = \frac{z^2-z}{\sqrt{2}(z^2+1)} \;//$$

② $\cos\left(\frac{n\pi}{2}+\theta\right)$

Let $f(n) = \cos\left(\frac{n\pi}{2}+\theta\right)$

$$f(n) = \cos\left(\frac{n\pi}{2}\right)\cos\theta - \sin\left(\frac{n\pi}{2}\right)\sin\theta$$

$$z[f(n)] = \cos\theta\, z\left[\cos\left(\frac{n\pi}{2}\right)\right] - \sin\theta\, z\left[\sin\left(\frac{n\pi}{2}\right)\right]$$

WKT

$$z[\cos n\theta] = \frac{z^2-z\cos\theta}{z^2-2z\cos\theta+1}$$

$$z\left[\cos\left(\frac{n\pi}{2}\right)\right] = \frac{z^2-z\cos(\pi/2)}{z^2-2z\cos(\pi/2)+1}$$

$$z\left[\cos\left(\frac{n\pi}{2}\right)\right] = \frac{z^2}{z^2+1}$$

$$Z[\sin(n\theta)] = \frac{z\sin\theta}{z^2-2z\cos\theta+1}$$

$$Z\left[\sin\left(\frac{n\pi}{2}\right)\right] = \frac{z\sin(\pi/2)}{z^2-2z\cos(\pi/2)+1}$$

$$= \frac{z}{z^2+1}$$

① $F(z) = \dfrac{z^2\cos\theta}{z^2+1} - \dfrac{z\sin\theta}{z^2+1}$

$$F(z) = \frac{z^2\cos\theta - z\sin\theta}{z^2+1} \quad //$$

③ $2n + \sin\left(\frac{n\pi}{2}\right)+1$

Let $f(n) = 2n + \sin\left(\frac{n\pi}{2}\right)+1$

$$Z[f(n)] = 2Z[n] = Z\left[\sin\left(\frac{n\pi}{2}\right)\right] + Z[1] \; - ①$$

$$Z(n) = \frac{z}{(z-1)^2} \Rightarrow \frac{2z}{(z-1)^2}$$

$$Z[\sin n\theta] = \frac{z\sin\theta}{z^2-2z\cos\theta+1}$$

$$Z\left[\sin\left(\frac{n\pi}{2}\right)\right] = \frac{z\sin(\pi/2)}{z^2-2z\cos(\pi/2)+1}$$

$$= \frac{z}{z^2+1}$$

$$Z[1] = \frac{z}{z-1}$$

$$Z[f(n)] = \frac{2z}{(z-1)^2} + \frac{z}{z^2+1} + \frac{z}{z-1} \quad //$$

④ $2n + \sin\left(\frac{n\pi}{4}\right)+1$

Let $f(n) = 2n + \sin\left(\frac{n\pi}{4}\right) + 1$

$$z[f(n)] = 2z[n] + z\left[\sin\left(\frac{n\pi}{4}\right)\right] + z[1] \quad -①$$

WKT $z[n] = \frac{z}{(z-1)^2}$

$$z[\sin n\theta] = \frac{z\sin\theta}{z^2 - 2z\cos\theta + 1}$$

$$z\left[\sin\left(\frac{n\pi}{4}\right)\right] = \frac{z\sin(\pi/4)}{z^2 - 2z\cos(\pi/4) + 1}$$

$$= \frac{z(1/\sqrt{2})}{z^2 - 2z\left(\frac{1}{\sqrt{2}}\right) + 1}$$

$$= \frac{\frac{z}{\sqrt{2}}}{\frac{\sqrt{2}z^2 - 2z + \sqrt{2}}{\sqrt{2}}}$$

$$z\left[\sin\left(\frac{n\pi}{4}\right)\right] = \frac{z}{\sqrt{2}z^2 - 2z + \sqrt{2}}$$

$$z[1] = \frac{z}{z-1}$$

$$① \Rightarrow F(z) = \frac{2z}{(z-1)^2} + \frac{z}{\sqrt{2}z^2 - 2z + \sqrt{2}} + \frac{z}{z-1}$$

⑤ $3^n \cos\left(\frac{n\pi}{4}\right)$

Let $f(n) = \cos\left(\frac{n\pi}{4}\right)$

$$z[f(n)] = z\left[\cos\left(\frac{n\pi}{4}\right)\right]$$

$$F(z) = \frac{z^2 - z\cos(\pi/4)}{z^2 - 2z\cos(\pi/4) + 1}$$

$$= \frac{z^2 - z\left(\frac{1}{\sqrt{2}}\right)}{z^2 - 2z\left(\frac{1}{\sqrt{2}}\right) + 1}$$

$$F(z) = \frac{\sqrt{2}z^2 - z}{\sqrt{2}z^2 - 2z + \sqrt{2}}$$

$$\text{WKT } z[a^n f(n)] = F\left(\frac{z}{a}\right)$$

$$z[3^n f(n)] = F\left(\frac{z}{3}\right)$$

$$z\left[3^n \cos\left(\frac{n\pi}{4}\right)\right] = \frac{\sqrt{2}\left(\frac{z}{3}\right)^2 - \left(\frac{z}{3}\right)}{\sqrt{2}\left(\frac{z}{3}\right)^2 - 2\left(\frac{z}{3}\right) + \sqrt{2}}$$

$$= \frac{\frac{\sqrt{2}}{9} z^2 - \frac{z}{3}}{\frac{\sqrt{2}}{9} z^2 - \frac{2z}{3} + \sqrt{2}}$$

$$= \frac{\cancel{\frac{1}{9}}(\sqrt{2} z^2 - 3z)}{\cancel{\frac{1}{9}}(\sqrt{2} z^2 - 6z + 9\sqrt{2})}$$

$$= \frac{\sqrt{2} z^2 - 3z}{\sqrt{2} z^2 - 6z + 9\sqrt{2}} \ //$$

6 $\sin(3n+5)$

Let $f(n) = \sin(3n+5)$

$$f(n) = \sin(3n)\cos 5 + \cos(3n)\sin 5$$

$$z[f(n)] = \cos 5 \ z[\sin(3n)] + \sin 5 \ z[\cos(3n)] \quad -(1)$$

$$z[\cos(n\theta)] = \frac{z^2 - z\cos\theta}{z^2 - 2z\cos\theta + 1}$$

$$z[\cos 3n] = \frac{z^2 - z\cos 3}{z^2 - 2z\cos 3 + 1}$$

$$z[\sin n\theta] = \frac{z\sin\theta}{z^2 - 2z\cos\theta + 1}$$

$$z[\sin 3] = \frac{z\sin 3}{z^2 - 2z\cos 3 + 1}$$

$$F(z) = \frac{z\sin 3\cos 5}{z^2-2z\cos 3+1} + \frac{(z^2-z\cos 3)\sin 3}{z^2-2z\cos 3+1}$$

$$= \frac{z\sin 3\cos 5 + z^2\sin 3 - z\sin 5\cos 3}{z^2-2z\cos 3+1}$$

$$= \frac{z[\sin 3\cos 5 - \sin 5\cos 3] + z^2\sin 3}{z^2-2z\cos 3+1}$$

$$= \frac{z\sin(-2) + z^2\sin 3}{z^2-2z\cos 3+1}$$

$$= \frac{z^2\sin 3 - z\sin 2}{z^2-2z\cos 3+1} \quad //$$

(7) $a^n \sin n\theta$

Let $f(n) = \sin n\theta$

$$Z[f(n)] = Z[\sin n\theta]$$

$$Z[\sin n\theta] = \frac{z\sin\theta}{z^2-2z\cos\theta+1}$$

WKT $Z[a^n f(n)] = F\left(\frac{z}{a}\right)$

$$Z[a^n \sin n\theta] = \frac{\frac{z}{a}\sin n\theta}{\left(\frac{z}{a}\right)^2 - 2\left(\frac{z}{a}\right)\cos\theta + 1}$$

$$= \frac{\frac{z}{a}\sin n\theta}{\frac{z^2-2az\cos\theta+a^2}{a^2}}$$

$$= \frac{az\sin n\theta}{z^2-2az\cos\theta+a^2} \quad //$$

(8) $\bar{a}^{n} \cos n\theta$

$$z[f(n)] = z[\cos n\theta]$$

$$F(z) = \frac{z^2 - z\cos\theta}{z^2 - 2z\cos\theta + 1}$$

WKT $z[\bar{a}^{n} f(n)] = F[az]$

$$z[\bar{a}^{n}\cos n\theta] = \frac{a^2z^2 - az\cos\theta}{a^2z^2 - 2az\cos\theta + 1} \,//$$

Inverse z-transforms:-

Suppose $F(z)$ be a z-transform of $f(x)$, then the inverse z-transform of $F(z)$ can be defined as

$$z^{-1}[F(z)] = f(n).$$

Standard Results:-

1. $z^{-1}\left[F\left(\frac{z}{a}\right)\right] = a^n f(n)$
2. $z^{-1}[F(az)] = \bar{a}^{n} f(n)$
3. $z^{-1}\left[\frac{z}{z-1}\right] = 1$
4. $z^{-1}\left[\frac{z}{z-a}\right] = a^n$
5. $z^{-1}\left[\frac{z}{(z-1)^2}\right] = n$
6. $z^{-1}\left[\frac{z^2+z}{(z-1)^3}\right] = n^2$
7. $z^{-1}\left[\frac{az}{(z-a)^2}\right] = a^n \cdot n$
8. $z^{-1}\left[\frac{z}{z+1}\right] = (-1)^n$
9. $z^{-1}\left[\frac{z^2}{z^2+1}\right] = \cos\left(\frac{n\pi}{2}\right)$
10. $z^{-1}\left[\frac{z}{z^2+1}\right] = \sin\left(\frac{n\pi}{2}\right)$

Find the inverse z-transform of the following :-

1. $\frac{z}{(z-2)(z-3)}$

Let $F(z) = \frac{z}{(z-2)(z-3)}$

$$\frac{F(z)}{z} = \frac{1}{(z-2)(z-3)}$$

$$\frac{1}{(z-2)(z-3)} = \frac{A}{(z-2)} + \frac{B}{(z-3)}$$

$1 = A(z-3) + B(z-2)$ — (2)

when $z = 2$

(2) $\Rightarrow 1 = A(2-3)$

$1 = -A$

$\boxed{A = -1}$

when $z = 3$

(2) $\Rightarrow 1 = B(3-2)$

$1 = B$

$\boxed{B = 1}$

(1) $\Rightarrow \frac{F(z)}{z} = \frac{-1}{z-2} + \frac{1}{z-3}$

$$F(z) = \frac{-z}{z-2} + \frac{z}{z-3}$$

$$Z^{-1}[F(z)] = -Z^{-1}\left[\frac{z}{z-2}\right] + Z^{-1}\left[\frac{z}{z-3}\right]$$

$$= -2^n + 3^n$$

$$f(n) = 3^n - 2^n$$

2. $\frac{z}{z^2+7z+10}$

$$F(z) = \frac{z}{z^2+7z+10}$$

$$\frac{F(z)}{z} = \frac{1}{z^2+7z+10}$$

$$\frac{F(z)}{z} = \frac{1}{z^2+2z+5z+10}$$

$$\frac{F(z)}{z} = \frac{1}{(z+2)(z+5)}$$

$$\frac{F(z)}{z} = \frac{A}{z+2} + \frac{B}{z+5}$$

$$\frac{1}{(z+2)(z+5)} = A(z+5) + B(z+2) \quad -① $$

when $z=-2$

$\Rightarrow ① \Rightarrow 1 = A(-2+5)$

$1 = A(3)$

$A = \frac{1}{3}$

when $z=-5$

$1 = B(-5+2)$

$1 = B(-3)$

$B = -\frac{1}{3}$

$$\frac{F(z)}{z} = \frac{1}{3}\frac{1}{z+2} - \frac{1}{3}\frac{1}{z+5}$$

$$z^{-1}[F(z)] = \frac{1}{3} z^{-1}\left[\frac{z}{z-(-2)}\right] - \frac{1}{3} z^{-1}\left[\frac{z}{z-(-5)}\right]$$

$$f(n) = \frac{1}{3}(-2)^n - \frac{1}{3}(-5)^n$$

$$f(n) = \frac{1}{3}\left[(-2)^n - (-5)^n\right] //$$

③ $\frac{3z^2+2z}{(5z-1)(5z+2)}$

Let $F(z) = \frac{3z^2+2z}{(5z-1)(5z+2)}$

$$\frac{F(z)}{z} = \frac{3z+2}{(5z-1)(5z+2)} \quad -①$$

$$\frac{3z+2}{(5z-1)(5z+2)} = \frac{A}{5z-1} + \frac{B}{5z+2}$$

$$3z+2 = A(5z+2) + B(5z-1) \quad -②$$

When $z = \frac{1}{5}$

$② \Rightarrow \frac{3}{5} + 2 = A(3)$

$\Rightarrow \frac{13}{5} = 3A \Rightarrow \boxed{A = \frac{13}{15}}$

when $z = -\frac{2}{5}$

$② \Rightarrow 3\left(-\frac{2}{5}\right) + 2 = B(-2-1)$

$\Rightarrow -\frac{6}{5} + 2 = -3B$

$\Rightarrow \frac{-6+10}{5} = -3B \quad \boxed{B = \frac{-4}{15}}$

$① \Rightarrow \frac{F(z)}{z} = \frac{13}{15}\frac{1}{(5z-1)} - \frac{4}{15}\cdot\frac{1}{(5z+2)}$

$$= \frac{13}{15\times5}\cdot\frac{z}{(z-\frac{1}{5})} - \frac{4}{15\times5}\frac{z}{(z+\frac{2}{5})}$$

$$F(z) = \frac{13}{75}\left[\frac{z}{z-\frac{1}{5}}\right] - \frac{4}{75}\left[\frac{z}{z-(-\frac{2}{5})}\right]$$

$$z^{-1}[F(z)] = \frac{13}{75}z^{-1}\left[\frac{z}{z-\frac{1}{5}}\right] - \frac{4}{75}z^{-1}\left[\frac{z}{z-(-\frac{2}{5})}\right]$$

$$f(n) = \frac{13}{75}\left(\frac{1}{5}\right)^n - \frac{4}{75}\left(\frac{-2}{5}\right)^n //$$

(4) $\frac{18z^2}{(2z-1)(4z+1)}$

Let $F(z) = \frac{18z^2}{(2z-1)(4z+1)}$

$$\frac{F(z)}{z} = \frac{18z}{(2z-1)(4z+1)} \quad -①$$

$$\frac{18z}{(2z-1)(4z+1)} = \frac{A}{(2z-1)} + \frac{B}{(4z+1)} \quad -②$$

$$18z = A(4z+1) + B(2z-1)$$

when $z = \frac{1}{2}$

$② \Rightarrow 9 = 3A \Rightarrow \boxed{A=3}$

when $z = -\frac{1}{4}$

$② \Rightarrow 18\left(-\frac{1}{4}\right) = B\left[2\left(-\frac{1}{4}\right)-1\right]$

$\Rightarrow \frac{-9}{2} = -\frac{3}{2}B$

$\boxed{B=3}$

$$① \Rightarrow F(z) = \frac{3}{(2z-1)} + \frac{3}{(4z+1)}$$

$$= \frac{3z}{(2z-1)} + \frac{3z}{(4z+1)}$$

$$= \frac{3}{2}\left[\frac{z}{z-\frac{1}{2}}\right] + \frac{3}{4}\left[\frac{z}{z+\frac{1}{4}}\right]$$

$$z^{-1}[F(z)] = \frac{3}{2}z^{-1}\left[\frac{z}{z-\frac{1}{2}}\right] + \frac{3}{4}z^{-1}\left[\frac{z}{z-(-\frac{1}{4})}\right]$$

$$f(n) = \frac{3}{2}\left(\frac{1}{2}\right)^n + \frac{3}{4}\left(\frac{-1}{4}\right)^n //$$

⑤ $\dfrac{2z^2+3z}{(z+2)(z-4)}$

$$F(z) = \frac{2z^2+3z}{(z+2)(z-4)}$$

$$\frac{F(z)}{z} = \frac{2z+3}{(z+2)(z-4)} \quad —①$$

$$\frac{2z+3}{(z+2)(z-4)} = \frac{A}{(z+2)} + \frac{B}{(z-4)}$$

$$2z+3 = A(z-4) + B(z+2) \quad —②$$

when $z=4$

② $\Rightarrow 11 = B(6)$

$B = \frac{11}{6}$

when $z=-2$

$-1 = A(-6)$

$A = \frac{1}{6}$

$$① \Rightarrow \frac{F(z)}{z} = \frac{1}{6}\frac{1}{(z+2)} + \frac{11}{6}\frac{1}{(z-4)}$$

$$F(z) = \frac{1}{6}\frac{z}{(z+2)} + \frac{11}{6}\frac{z}{(z-4)}$$

$$z^{-1}[F(z)] = \frac{1}{6} z^{-1}\left[\frac{z}{z+2}\right] + \frac{11}{6} z^{-1}\left[\frac{z}{z-4}\right]$$

$$f(n) = \frac{1}{6}(-2)^n + \frac{11}{6}(4)^n \,//$$

⑥ $\dfrac{8z^2}{(2z-1)(4z-1)}$

$$F(z) = \frac{8z^2}{(2z-1)(4z-1)}$$

$$\frac{F(z)}{z} = \frac{8z}{(2z-1)(4z-1)} \quad —①$$

$$\frac{8z}{(2z-1)(4z-1)} = \frac{A}{(2z-1)} + \frac{B}{(4z-1)}$$

$$8z = A(4z-1) + B(2z-1) \quad —②$$

when $z = \frac{1}{4}$

$$② \Rightarrow 2 = B\left(2\left(\tfrac{1}{4}\right)-1\right)$$

$\Rightarrow 2 = B\left(\frac{1}{2}-1\right)$ when $z = \frac{1}{2}$

$2 = B\left(\frac{1-2}{2}\right)$ ② $\Rightarrow 4 = A(2-1)$

$\Rightarrow 4 = A$

$B = -4$ $A = 4$

$$① \Rightarrow \frac{F(z)}{z} = \frac{4}{(2z-1)} - \frac{4}{(4z-1)}$$

$$F(z) = 4\frac{z}{(2z-1)} - 4\cdot\frac{z}{(4z-1)}$$

$$z^{-1}[F(z)] = 4z^{-1}\left[\frac{z}{2z-1}\right] - 4z^{-1}\left[\frac{z}{4z-1}\right]$$

$$= \frac{\cancel{4}^{2}}{\cancel{2}} z^{-1}\left[\frac{z}{z-\frac{1}{2}}\right] - \frac{4}{4} z^{-1}\left[\frac{z}{z-\frac{1}{4}}\right]$$

$$f(n) = 2\left(\frac{1}{2}\right)^n - \left(\frac{1}{4}\right)^n //$$

7) $\frac{4z^2-2z}{z^3-5z^2+8z-4}$

$$F(z) = \frac{4z^2-2z}{z^3-5z^2+8z-4}$$

$$\frac{F(z)}{z} = \frac{4z-2}{z^3-5z^2+8z-4}$$

$$= \frac{4z-2}{z^3-5z^2+8z-4}$$

$$= \frac{4z-2}{(z-1)(z-2)^2}$$

$$\frac{4z-2}{(z-1)(z-2)^2} = \frac{A}{z-1} + \frac{B}{z-2} + \frac{C}{(z-2)^2}$$

$$4z-2 = A(z-2)^2 + B(z-1)(z-2) + C(z-1) \quad -②$$

when $z=1$ when $z = 2$

$2 = A$ $6 = C$

$\boxed{A = 2}$ $\boxed{C = 6}$

$A + B = 0$

$B = -A$

$\boxed{B = -2}$

① $\Rightarrow \dfrac{F(z)}{z} = \dfrac{2}{z-1} - \dfrac{2}{z-2} + \dfrac{6}{(z-2)^2}$

$$= 2\,\frac{z}{z-1} - 2\,\frac{z}{z-2} + 3\,\frac{2z}{(z-2)^2}$$

$$z^{-1}[F(z)] = 2z^{-1}\left[\frac{z}{z-1}\right] - 2z^{-1}\left[\frac{z}{z-2}\right] + 3z^{-1}\left[\frac{2z}{(z-2)^2}\right]$$

$$= 2(1) - 2(2^n) + 3.n2^n \;//$$

Difference Equations:-

Step ①:- Express the given difference equation in the notation of $y_n, y_{n+1}, y_{n+2} \ldots$

Step ②:- Apply z-transform on both sides and substitute

$$z[y_{n+2}] = z^2\left[\bar{y}(z) - y_0 - \frac{y_1}{z}\right]$$

$$z[y_{n+1}] = z\left[\bar{y}(z) - y_0\right]$$

$$z[y_n] = \bar{y}(z)$$

Step ③:- write $\bar{y}(z)$ has a function of z, hence apply the inverse z-transform and find $y(n)$.

① Solve the difference equation using z-transform $y_{n+2} - 4y_n = 0$; Subject to the conditions $y_0 = 0, y_1 = 2$.

Given $y_{n+2} - 4y_n = 0 \quad y_0 = 0, \; y_1 = 2$

$$\Rightarrow z[y_{n+2}] - 4z[y_n] = z[0]$$

$$\Rightarrow z^2\left[\bar{y}(z) - y_0 - \frac{y_1}{z}\right] - 4\bar{y}(z) = 0$$

$$\Rightarrow z^2\left[\bar{y}(z) - 0 - \frac{2}{z}\right] - 4\bar{y}(z) = 0$$

$$\Rightarrow z^2\bar{y}(z) - 2z - 4\bar{y}(z) = 0$$

$$(z^2 - 4)\bar{y}(z) = 2z$$

$$\Rightarrow \bar{y}(z) = \frac{2z}{(z^2-4)}$$

$$\bar{y}(z) = \frac{2z}{(z+2)(z-2)}$$

$$\frac{\bar{y}(z)}{z} = \frac{2}{(z-2)(z+2)} \quad \text{—①}$$

$$\frac{z}{(z-2)(z+2)} = \frac{A}{(z-2)} + \frac{B}{(z+2)}$$

$$2 = A(z+2) + B(z-2) \quad \text{—②}$$

when $z=2$

② $\Rightarrow 2 = 4A$

$A = \frac{1}{2}$

when $z=-2$

② $\Rightarrow 2 = -4B$

$B = -\frac{1}{2}$

$$① \Rightarrow \frac{\bar{y}(z)}{z} = \frac{1}{2}\cdot\frac{1}{z-2} - \frac{1}{2}\,\frac{1}{z+2}$$

$$\bar{y}(z) = \frac{1}{2}\,\frac{z}{z-2} - \frac{1}{2}\,\frac{z}{z+2}$$

$$z^{-1}[\bar{y}(z)] = \frac{1}{2}\, z^{-1}\left[\frac{z}{z-2}\right] - \frac{1}{2}\, z^{-1}\left[\frac{z}{z+2}\right]$$

$$y(n) = \frac{1}{2}(2)^n - \frac{1}{2}(-2)^n$$

② Using z-transform solve the difference equation $y_{n+2} + 6y_{n+1} + 9y_n = 2^n$, subject to the conditions $y_0 = 0$, $y_1 = 0$.

Given $y_{n+2} + 6y_{n+1} + 9y_n = 2^n$, $y_0 = 0$, $y_1 = 0$

$$z[y_{n+2}] + 6z[y_{n+1}] + 9z[y_n] = z[2^n]$$

$$z^2\left[\bar{y}(z) - y_0 - \frac{y_1}{z}\right] + 6z[\bar{y}(z) - y_0] + 9\bar{y}(z) = \frac{z}{z-2}$$

$$z^2\bar{y}(z) + 6z\bar{y}(z) + 9\bar{y}(z) = \frac{z}{z-2}$$

$$(z^2+6z+9)\,\bar{y}(z) = \frac{z}{z-2}$$

$$\bar{y}(z) = \frac{z}{(z-2)(z+3)^2}$$

$$\frac{\bar{y}(z)}{z} = \frac{1}{(z-2)(z+3)^2} \quad -①$$

$$\frac{1}{(z-2)(z+3)^2} = \frac{A}{z-2} + \frac{B}{z+3} + \frac{C}{(z+3)^2}$$

$$1 = A(z+3)^2 + B(z-2)(z+3) + C(z-2) \quad -②$$

when $z = 2$

$② \Rightarrow 1 = A(2+3)^2$

$1 = 25A$

$A = \frac{1}{25}$

when $z = -3$

$② \Rightarrow 1 = C(-5)$

$C = \frac{-1}{5}$

when

$A + B = 0$

$B = -A$

$B = \frac{-1}{25}$

$$① \Rightarrow \frac{\bar{y}(z)}{z} = \frac{1}{25} \cdot \frac{1}{z-2} - \frac{1}{25} \frac{1}{z+3} - \frac{1}{5} \cdot \frac{1}{(z+3)^2}$$

$$\bar{y}(z) = \frac{1}{25} \frac{z}{z-2} - \frac{1}{25} \frac{z}{z+3} - \frac{1}{5} \cdot \frac{z}{(z+3)^2}$$

$$\bar{y}(z) = \frac{1}{25} \cdot \frac{z}{z-2} - \frac{1}{25} \cdot \frac{z}{z-(-3)} + \frac{1}{5} \frac{(-3z)}{(z-(-3))^2}$$

$$z^{-1}[\bar{y}(z)] = \frac{1}{25} z^{-1}\left[\frac{z}{z-2}\right] - \frac{1}{25} z^{-1}\left[\frac{z}{z-(-3)}\right] + \frac{1}{15} \frac{(-3z)}{(z-(-3))^2}$$

$$\Rightarrow y(n) = \frac{1}{25}(2)^n - \frac{1}{25}(-3)^n + \frac{1}{15}(-3)^n \cdot n \;//$$

③ Solve $u_{n+2} - 3u_{n+1} + 2u_n = 2^n$. Given $u_0 = 0$, $u_1 = 1$ by using Z-transform.

Given $u_{n+2} - 3u_{n+1} + 2u_n = 2^n$, $u_0 = 0$, $u_1 = 1$

$$\Rightarrow z[u_{n+2}] - 3z[u_{n+1}] + 2z[u_n] = z[2^n]$$

$$\Rightarrow z^2\left[\bar{u}(z) - u_0 - \frac{u_1}{z}\right] - 3z[\bar{u}(z) - u_0] + 2\bar{u}(z) = \frac{z}{z-2}$$

$$\Rightarrow z^2\bar{u}(z) - z - 3z\bar{u}(z) + 2\bar{u}(z) = \frac{z}{z-2}$$

$$(z^2 - 3z + 2)\bar{u}(z) = \frac{z}{z-2} + z$$

$$(z-1)(z-2)\bar{u}(z) = \frac{z + z(z-2)}{z-2}$$

$$(z-1)(z-2)\bar{u}(z) = \frac{z^2 - z}{z-2}$$

$$\bar{u}(z) = \frac{z^2 - z}{(z-1)(z-2)^2}$$

$$\bar{u}(z) = \frac{z(z-1)}{(z-1)(z-2)^2}$$

$$\bar{u}(z) = \frac{z}{(z-2)^2}$$

$$\bar{u}(z) = \frac{1}{2} \cdot \frac{2z}{(z-2)^2}$$

$$z^{-1}[\bar{u}(z)] = \frac{1}{2} z^{-1}\left[\frac{2z}{(z-2)^2}\right]$$

$$u(n) = \frac{1}{2} n 2^n$$

$$u(n) = 2^{n-1} n \quad //$$

(4) Solve the difference equation $u_{n+2} + 2u_{n+1} + u_n = n$, subject to the conditions $u_0 = 0$, $u_1 = 0$.

Given $u_{n+2} + 2u_{n+1} + u_n = n$, $u_0 = 0$, $u_1 = 0$

$$\Rightarrow z[u_{n+2}] + 2z[u_{n+1}] + z[u_n] = z[n]$$

$$\Rightarrow z^2\left[\bar{u}(z) - u_0 - \frac{u_1}{z}\right] + 2z[\bar{u}(z)] + \bar{u}(z) = \frac{z}{(z-1)^2}$$

$$z^2 \bar{u}(z) + 2z\bar{u}(z) + \bar{u}(z) = \frac{z}{(z-1)^2}$$

$$(z^2 + 2z + 1)\bar{u}(z) = \frac{z}{(z-1)^2}$$

$$(z+1)^2 \bar{u}(z) = \frac{z}{(z-1)^2}$$

$$\bar{u}(z) = \frac{z}{(z-1)^2(z+1)^2}$$

$$\frac{\bar{u}(z)}{z} = \frac{1}{(z-1)^2(z+1)^2} \quad \text{—(1)}$$

$$\frac{1}{(z-1)^2(z+1)^2} = \frac{A}{z-1} + \frac{B}{(z-1)^2} + \frac{C}{(z+1)} + \frac{D}{(z+1)^2}$$

$$\Rightarrow 1 = A(z-1)(z+1)^2 + B(z+1)^2 + C(z+1)(z-1)^2 + D(z-1)^2 \quad \text{—(2)}$$

when $z=1$

$② \Rightarrow 1 = 4B$

$B = \frac{1}{4}$

when $z=-1$

$② \Rightarrow 1 = 4D$

$D = \frac{1}{4}$

$-A+B+C+D=1$

$-A+\frac{1}{4}+C+\frac{1}{4}=1$

$-A+C=\frac{1}{2}$ — ③

when $-A+C=0$ — ④

$③+④ \Rightarrow 2C=\frac{1}{2}$

$C=\frac{1}{4}$

$A=-\frac{1}{4}$

$$① \Rightarrow \frac{\bar{u}(z)}{z} = -\frac{1}{4}\frac{1}{z-1} + \frac{1}{4}\frac{1}{(z-1)^2} + \frac{1}{4}\cdot\frac{1}{(z+1)} + \frac{1}{4}\frac{1}{(z+1)^2}$$

$$\Rightarrow \bar{u}(z) = -\frac{1}{4}\frac{z}{z-1} + \frac{1}{4}\frac{z}{(z-1)^2} + \frac{1}{4}\frac{z}{(z+1)} + \frac{1}{4}\frac{z}{(z+1)^2}$$

$$z^{-1}[\bar{u}(z)] = -\frac{1}{4}z^{-1}\left[\frac{z}{z-1}\right] + \frac{1}{4}z^{-1}\left[\frac{z}{(z-1)^2}\right] + \frac{1}{4}z^{-1}\left[\frac{z}{z-(-1)}\right] + \frac{1}{4}\left[\frac{(-1)z}{z-(-1)^2}\right]$$

$$u(n) = -\frac{1}{4} + \frac{1}{4}n + \frac{1}{4}(-1)^n + \frac{1}{4}(-1)^n \cdot n \ //$$

⑤ Solve the difference equation $u_{n+2}+4u_{n+1}+3u_n=3^n$, subject to the conditions $u_0=0,\ u_1=1$

Given $u_{n+2}+4u_{n+1}+3u_n=3^n,\ u_0=0,\ u_1=1$

$$\Rightarrow z[u_{n+2}] + 4z[u_{n+1}] + 3z[u_n] = z[3^n]$$

$$\Rightarrow z^2\left[\bar{u}(z)-u_0-\frac{u_1}{z}\right] + 4z[\bar{u}(z)-u_0] + 3\bar{u}(z) = \frac{z}{z-3}$$

$$z^2\bar{u}(z) - z + 4z\bar{u}(z) + 3\bar{u}(z) = \frac{z}{z-3}$$

$$[z^2+4z+3]\bar{u}(z) = \frac{z}{z-3} + z$$

$$(z+3)(z+1)\bar{u}(z) = \frac{z+z(z-3)}{z-3}$$

$$\bar{u}(z) = \frac{z+z^2-3z}{(z+1)(z-3)(z+3)}$$

$$\bar{u}(z) = \frac{z^2-2z}{(z+1)(z-3)(z+3)}$$

$$\frac{\bar{u}(z)}{z} = \frac{z-2}{(z+1)(z+3)(z-3)} \quad — ①$$

$$\frac{z-2}{(z+1)(z+3)(z-3)} = \frac{A}{z+1} + \frac{B}{z+3} + \frac{C}{z-3}$$

$$z-2 = A(z+3)(z-3) + B(z+1)(z-3) + C(z+1)(z+3) \quad -(2)$$

when $z=-3$ $\qquad$ $z=3$ $\qquad$ $z=-1$

(2) $\Rightarrow -5 = B(-2)(-6)$ $\qquad$ (2) $\Rightarrow 1 = C(4)(6)$ $\qquad$ (2) $\Rightarrow -3 = A(2)(-4)$

$-5 = 12B$ $\qquad$ $1 = 24C$ $\qquad$ $-3 = -8A$

$B = \frac{-5}{12}$ $\qquad$ $C = \frac{1}{24}$ $\qquad$ $A = \frac{3}{8}$

$$(1) \Rightarrow \frac{\bar{u}(z)}{z} = \frac{3}{8}\frac{1}{z+1} - \frac{5}{12}\frac{1}{z+3} + \frac{1}{24}\frac{1}{z-3}$$

$$\bar{u}(z) = \frac{3}{8}\frac{z}{z+1} - \frac{5}{12}\cdot\frac{z}{z+3} + \frac{1}{24}\frac{z}{z-3}$$

$$z^{-1}[\bar{u}(z)] = \frac{3}{8} z^{-1}\left[\frac{z}{z+1}\right] - \frac{5}{12} z^{-1}\left[\frac{z}{z+3}\right] + \frac{1}{24} z^{-1}\left[\frac{z}{z-3}\right]$$

$$u(n) = \frac{3}{8}(-1)^n - \frac{5}{12}(-3)^n + \frac{1}{24}(3)^n \,//$$

(6) Solve the difference equation $u_{n+2} - 3u_{n+1} + 2u_n = 3^n$, subject to the conditions $u_0 = u_1 = 0$.

Given $u_{n+2} - 3u_{n+1} + 2u_n = 3^n$, $u_0 = 0$, $u_1 = 0$

$$\Rightarrow z[u_{n+2}] - 3z[u_{n+1}] + 2z[u_n] = z[3^n]$$

$$\Rightarrow z^2\left[\bar{u}(z) - u_0 - \frac{u_1}{z}\right] - 3z[\bar{u}(z) - u_0] + 2\bar{u}(z) = \frac{z}{z-3}$$

$$z^2\bar{u}(z) - 3z\bar{u}(z) + 2\bar{u}(z) = \frac{z}{z-3}$$

$$[z^2 - 3z + 2]\bar{u}(z) = \frac{z}{z-3}$$

$$(z-1)(z-2)\bar{u}(z) = \frac{z}{z-3}$$

$$\bar{u}(z) = \frac{z}{(z-1)(z-2)(z-3)} \quad -(1)$$

$$\frac{\bar{u}(z)}{z} = \frac{1}{(z-1)(z-2)(z-3)}$$

$$\frac{1}{(z-1)(z-2)(z-3)} = \frac{A}{z-1} + \frac{B}{z-2} + \frac{C}{z-3}$$

$$1 = A(z-2)(z-3) + B(z-1)(z-3) + C(z-1)(z-2) \quad \text{--- (2)}$$

when $z=2$

(2) $\Rightarrow 1 = B(1)(1)$

$\boxed{B=1}$

$z=1$

$1 = A(-1)(-2)$

$A = 2$

$z=3$

$1 = C(2)(1)$

$C = \frac{1}{2}$

$$\frac{\bar{u}(z)}{z} = \frac{2}{(z-1)} + \frac{1}{(z-2)} + \frac{1}{2}\left(\frac{1}{z-3}\right)$$

$$\bar{u}(z) = 2\left(\frac{z}{z-1}\right) + \frac{z}{z-2} + \frac{1}{2}\left(\frac{z}{z-3}\right)$$

$$z^{-1}[\bar{u}(z)] = 2z^{-1}\left[\frac{z}{z-1}\right] + z^{-1}\left[\frac{z}{z-2}\right] + \frac{1}{2}z^{-1}\left[\frac{z}{z-3}\right]$$

$$u(n) = 2(1) + (2)^n + \frac{1}{2}(3^n) \quad //$$